Chapter 16
Characterization of Domains of Fractional Powers

This chapter is devoted to characterizing the domains of fractional powers $A^\theta$ for a sectorial operator $A$ in a Hilbert or Banach space. However, the situations are quite different in Hilbert and Banach space cases. Anyway, in both cases, the boundedness of the $H_\infty$ functional calculus for $A$ plays an important role for the complete characterization of $\mathcal{D}(A^\theta)$. In the Hilbert space case, the $H_\infty$ functional calculus for sectorial operators was invented by McIntosh in 1986 and was generalized in the Banach space case by Cowling–Doust–McIntosh–Yagi in 1996. In the Hilbert space case, Theorem 2.30 due to Kato ensures that any sectorial operator associated with a sesquilinear form has a bounded $H_\infty$ functional calculus. To the contrary, in the Banach space case, there is no convenient sufficient condition ensuring the boundedness of the functional calculus. Therefore, for elliptic differential operators in $L_p(\Omega)$, $1 < p < \infty$, $p \neq 2$, equipped with suitable boundary conditions, we have to appeal to the theory of harmonic analysis or the theory of integral operators in order to prove the boundedness by some specific methods.

On the other hand, we may notice that estimation of the domain $\mathcal{D}(A^\theta)$ from interior or exterior is considerably easy as we have already done in Theorems 2.24 and 2.25 and will do in Remark 16.6. In our abstract results, we have repeatedly used the fractional powers of coefficient linear operators in abstract evolution equations for describing our structural assumptions such as (3.29), (3.30), (4.2), (4.21), (5.4), (5.5), and (5.6). For verifying these assumptions, however, their complete characterization is not necessarily required, but their appropriate estimation is often sufficient.

1 Domains of Fractional Powers in Hilbert Spaces

1.1 Case of Self-Adjoint Operators

Let $X$ be a Hilbert space. When $A$ is a positive definite self-adjoint operator of $X$, the domains of its fractional powers coincide with the interpolation spaces.

Theorem 16.1 Let $A$ be a positive definite self-adjoint operator of $X$. For any $0 < \theta < 1$, $[X, \mathcal{D}(A)]_{\theta} = \mathcal{D}(A^\theta)$ with isometry.

Proof We first recall that $A^{-z}F$ is an $X$-valued analytic function for $\text{Re } z > 0$ and is a continuous function for $\text{Re } z \geq 0$ for each $F \in X$ with the estimate $\|A^{-z}\| \leq 1$ for $\text{Re } z \geq 0$, see Remark 2.7.

Let $U \in \mathcal{D}(A^\theta)$. We consider the analytic function $F(z) = A^{-z}A^\theta U$. Then, $F \in \mathcal{H}(X, \mathcal{D}(A))$ with $U = F(0)$, where $\mathcal{H}(X, \mathcal{D}(A))$ denotes the function space defined in Chap. 1, Sect. 5.1. Therefore, $U \in [X, \mathcal{D}(A)]_{\theta}$, and

$$\|U\|_{[X, \mathcal{D}(A)]_{\theta}} \leq \|F\|_{\mathcal{H}(X, \mathcal{D}(A))} \leq \left( \sup_{-\infty < y < \infty} \|A^{iy}\| \right) \|A^\theta U\| = \|A^\theta U\|.$$ 

Conversely, let $U \in [X, \mathcal{D}(A)]_{\theta}$. By definition, there exists a function $F \in \mathcal{H}(X, \mathcal{D}(A))$ such that $U = F(0)$. We then introduce a function of the form

$$\Phi(z) = e^{\varepsilon(z-\theta)^2} (A^{z-1}F(z), AV), \quad \varepsilon > 0, \ V \in \mathcal{D}(A).$$

Obviously, $\Phi(z)$ is holomorphic in the strip $S = \{z \in \mathbb{C}; 0 < \text{Re } z < 1\}$ and is continuous in the closure $\overline{S}$. Furthermore, it is observed that $\lim_{|\text{Im } z| \to \infty} \Phi(z) = 0$ because of $|e^{\varepsilon(z-\theta)^2}| \leq e^{\varepsilon(1-|\text{Im } z|^2)}$. Therefore,

$$\sup_{z \in S} |\Phi(z)| \leq \max \left\{ \sup_{-\infty < y < \infty} |\Phi(iy)|, \sup_{-\infty < y < \infty} |\Phi(1 + iy)| \right\} \leq \left( \sup_{-\infty < y < \infty} \|A^{iy}\| \right) \varepsilon \|F\| \|\mathcal{H}(X, \mathcal{D}(A))\|.$$

Taking $z = \theta$, we obtain that

$$\|(A^{\theta-1}U, AV)\| \leq e^{\varepsilon} \|F\| \|\mathcal{H}(X, \mathcal{D}(A))\|, \quad \varepsilon > 0, \ V \in \mathcal{D}(A).$$

This means that $U \in \mathcal{D}(A^\theta)$ with the estimate $\|A^\theta U\| \leq e^{\varepsilon} \|F\| \|\mathcal{H}(X, \mathcal{D}(A))\|$. Taking the infimum of $\|F\| \|\mathcal{H}(X, \mathcal{D}(A))\|$ for all possible $F$ and afterwards tending $\varepsilon \to 0$, we observe that $\|A^\theta U\| \leq \|U\|_{[X, \mathcal{D}(A)]_{\theta}}$.

We hence verified that $\mathcal{D}(A^\theta) = [X, \mathcal{D}(A)]_{\theta}$ with the isometry

$$\|A^\theta U\| = \|U\|_{[X, \mathcal{D}(A)]_{\theta}}, \quad U \in \mathcal{D}(A^\theta).$$

We can in addition verify that $A$ satisfies the following square function estimate. Let $A \geq \delta$ and let $\{E(\mu)\}_{\delta \leq \mu < \infty}$ be the spectral resolution of $A$. Let $\Gamma_\omega: \lambda = \rho e^{\pm i\omega}, 0 \leq \rho < \infty$, where $0 < \omega \leq \pi$, and let $0 < \theta < 1$. Then, for $F \in X$,

$$\int_{\Gamma_\omega} \lambda^{2\theta-1} \|A^{-\theta}(\lambda - A)^{-1}F\|^2 d\lambda = \int_{\Gamma_\omega} \int_0^\infty \frac{|\lambda|^{2\theta-1} \mu^{2(1-\theta)}}{|\lambda - \mu|^2} d\|E(\mu)F\|^2 d\lambda$$

$$= \sum_{k=\pm 1} \int_0^\infty \int_0^\infty \frac{\rho^{2\theta-1} \mu^{2(1-\theta)}}{|\rho e^{ik\omega} - \mu|^2} d\rho d\|E(\mu)F\|^2$$