Chapter 4
Finite Difference Schemes for Convection-Diffusion Problems

This chapter is concerned with finite-difference discretisations of the stationary linear convection-diffusion problem

\[ \mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1, \quad (4.1) \]

with \( b \geq \beta > 0 \) on \([0,1]\). For the sake of simplicity we shall assume that

\[ c \geq 0 \quad \text{and} \quad b' \geq 0 \quad \text{on} \quad [0,1]. \quad (4.2) \]

Using (4.1) as a model problem, a general convergence theory for certain first- and second-order upwinded difference schemes on arbitrary and on layer-adapted meshes is derived. The close relationship between the differential operator and its upwinded discretisations is highlighted.

4.1 Notation

Meshes and mesh functions

Throughout this chapter let \( \bar{\omega} := 0 = x_0 < x_1 < \cdots < x_N = 1 \) be an arbitrary partition of \([0,1]\) with mesh intervals \( I_i := [x_{i-1}, x_i] \). The set of inner mesh points is denoted by \( \omega \). The midpoint of \( I_i \) is \( x_{i-1/2} := (x_i - x_{i+1})/2 \) and its length \( h_i := x_i - x_{i-1} \). Let \( h := \max_{i=1,...,N} h_i \) be the maximal mesh size.

We shall identify mesh functions \( v : \bar{\omega} \to \mathbb{R} : x_i \mapsto v_i \) with vectors \( v \in \mathbb{R}^{N+1} \) and with spline functions

\[ v \in V^\omega := \mathcal{S}_1^0(\bar{\omega}) := \left\{ w \in C^0[0,1] : w|_{I_i} \in \Pi_1 \quad \text{for} \quad i = 1, \ldots, N \right\}. \]

Let \( \mathbb{R}_0^{N+1} := \left\{ v \in \mathbb{R}^{N+1} : v_0 = v_N = 0 \right\} \) be the space of mesh functions that vanish at the boundary. Furthermore, \( V_0^\omega := \mathcal{S}_1^0(\bar{\omega}) \cap H_0^1(0,1) \).
Difference operators

In our notation of difference operators we follow Samarski’s text book [146]. For any mesh function \( v \in \mathbb{R}^{N+1} \) set

\[
\begin{align*}
v_{x;i} := & \frac{v_{i+1} - v_i}{h_{i+1}}, & v_{\bar{x};i} := & \frac{v_i - v_{i-1}}{h_i}, & v_{\hat{x};i} := & \frac{v_i - v_{i-1}}{2h_i}, \\
v_{\bar{x};i} := & \frac{v_{i+1} - v_i}{h_i}, & v_{\check{x};i} := & \frac{v_i - v_{i-1}}{h_i}, & v_{\ddot{x};i} := & \frac{v_{i+1} - v_i}{h_i}.
\end{align*}
\]

with the weighted mesh increment \( \bar{h} \) defined by

\[
\bar{h}_0 := \frac{h_1}{2}, \quad \bar{h}_i := \frac{h_i + h_{i+1}}{2}, \quad i = 1, \ldots, N - 1, \quad \text{and} \quad \bar{h}_N := \frac{h_N}{2}.
\]

To simplify the notation we set \( g_i := g(x_i) \) for any \( g \in C[0, 1] \).

Further, less frequently used, difference operators will be introduced when needed.

Discrete norms and inner products

For any mesh function \( v \in \mathbb{R}^{N+1} \) define the \( \ell_\infty \) (semi-)norms

\[
\|v\|_{\infty,\bar{\omega}} := \max_{i=1,\ldots,N-1} |v_i|, \quad \|v\|_{\infty,\bar{\omega}} := \max_{i=0,\ldots,N} |v_i|,
\]

\[
\|v\|_{\infty,\bar{\omega}} := \max_{i=0,\ldots,N-1} |v_i|, \quad ||v||_{\infty,\bar{\omega}} := \max \left\{ \varepsilon \|v_x\|_{\infty,\bar{\omega}}, \beta \|v\|_{\infty,\bar{\omega}} \right\},
\]

the \( \ell_1 \) norm

\[
\|v\|_{1,\bar{\omega}} := \sum_{j=0}^{N-1} h_{j+1} |v_j|
\]

and the \( w^{-1,\infty} \) norm

\[
\|v\|_{-1,\infty,\bar{\omega}} := \min_{V: V_x = v} \|V\|_{\infty,\bar{\omega}} = \min_{c \in \mathbb{R}} \left\| \sum_{j=1}^{N-1} h_{j+1} v_j + c \right\|_{\infty,\bar{\omega}}.
\]

We shall also use the following discrete inner products:

\[
[w, v]_{\bar{\omega}} := \sum_{i=0}^{N-1} h_{i+1} w_i v_i \quad \text{and} \quad (w, v)_{\bar{\omega}} := \sum_{i=1}^{N-1} h_{i+1} w_i v_i.
\]