12. Numerical solution of the Euler equations in general domains

12.1 Introduction

The purpose of this chapter is to generalize the methods described in Chap. 10 for computing inviscid compressible flows in one dimension to more dimensions and general domains. Although no new principles will emerge, this generalization is not completely straightforward, except for the JST scheme (which is its great charm), which will therefore not be discussed here.

We will mostly discuss the three-dimensional case, leaving the two-dimensional case for the reader to work out. The domain $\Omega \subset \mathbb{R}^3$ is arbitrary, and is assumed to contain a single block boundary-fitted structured grid with hexahedrons as cells, as discussed in Chap. 11. In practice the shape of the domain is often so complicated, that domain decomposition (cf. Sect. 11.3) has to be used. Domain decomposition is also frequently used to create subtasks for parallel processing. Conditions to couple subdomain solutions and iterative methods to iterate to global convergence have to be devised. Introductions to the mathematical background of domain decomposition are given in Chan and Mathew (1994) and Smith, Bjørstad, and Gropp (1996). We will not discuss domain decomposition.

For reasons to be explained in Chap. 14, the methods to be discussed in the present chapter do not work well in the (almost) incompressible case. A unified method for the compressible and the incompressible case will be presented in Chap. 14.

12.2 Analytic aspects

This section runs parallel to Sect. 10.2; here we discuss only extensions brought about by multi-dimensionality.

Euler equations

The Euler equations in Cartesian coordinates (1.76)–(1.78) can be written as
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\[ U_t + \partial f^\alpha(U)/\partial x^\alpha = Q, \quad x \in \Omega, \quad t \in (0, T), \tag{12.1} \]

where we sum over \( \alpha \in \{1, 2, 3\} \), and where

\[ U = \begin{pmatrix} m^1 \\ m^2 \\ m^3 \\ \rho E \end{pmatrix}, \quad f^\alpha = \begin{pmatrix} m^\alpha \\ u^\alpha m^1 + p\delta^\alpha_1 \\ u^\alpha m^2 + p\delta^\alpha_2 \\ u^\alpha m^3 + p\delta^\alpha_3 \\ m^\alpha H \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ \rho f^1_b \\ \rho f^2_b \\ \rho f^3_b \\ \rho q + \rho u^\beta f^\beta_b \end{pmatrix}, \tag{12.2} \]

where \( m^\alpha = \rho u^\alpha \), and \( \delta^\alpha_\beta \) is the Kronecker delta. This is the conservative formulation. We will also need the nonconservative formulation with dependent variables \( \rho, u \) and \( p \). For the momentum and energy equations we take the inviscid and nondiffusive versions of (1.27) and (1.43), respectively. We use the perfect gas law \( e = \frac{1}{\gamma - 1} \frac{p}{\rho} \) to eliminate \( e \) from the energy equation, which becomes:

\[ \frac{Dp}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt} = (1 - \gamma) p \text{div} u + (\gamma - 1) \rho q. \]

Using the mass conservation equation, \( D\rho/Dt \) is replaced by \(-\rho \text{div} u\). The nonconservative formulation of the Euler equations becomes:

\[ \frac{D\rho}{Dt} + \rho \text{div} u = 0, \tag{12.3} \]

\[ \frac{Du}{Dt} + \frac{1}{\rho} \text{grad} p = f_b, \tag{12.4} \]

\[ \frac{Dp}{Dt} + \gamma \rho \text{div} u = (\gamma - 1) \rho q. \tag{12.5} \]

**Homogeneity of the flux function**

Suppose the equation of state is given by

\[ p = \rho g(e). \]

Note that the perfect gas law is a special case. Since \( e = e(U) = \{ U_5 - \frac{1}{2} (U_2^2 + U_3^2 + U_4^2)/U_1 \}/U_1 \), we have \( e(\mu U) = e(U) \), with \( \mu \) an arbitrary real number. In the same way as in Sect. 10.2 it is easily shown that the flux functions are homogeneous of degree one:

\[ f^\alpha(\mu U) = \mu f^\alpha(U), \]

and that we have

\[ f^\alpha(U) = F^\alpha(U)U, \quad F^\alpha \equiv \partial f^\alpha(U)/\partial U. \]