V.8 Convergence for Nonlinear Problems

In Sect. V.6 we have seen a convergence result for one-leg methods (Theorem 6.10) applied to nonlinear problems satisfying a one-sided Lipschitz condition. An extension to linear multistep methods has been given in Theorem 6.11. A different and direct proof of this result will be the first goal of this section. Unfortunately, such a result is valid only for $A$-stable methods (whose order cannot exceed two). The subsequent parts of this section are then devoted to convergence results for nonlinear problems, where the assumptions on the method are relaxed (e.g., $A(\alpha)$-stability), but the class of problems considered is restricted. We shall present two different theories: the multiplier technique of Nevanlinna & Odeh (1981) and Lubich’s perturbation approach via the discrete variation of constants formula (Lubich 1991).

Problems Satisfying a One-Sided Lipschitz Condition

Suppose that the differential equation $y' = f(x, y)$ satisfies

$$\text{Re} \langle f(x, y) - f(x, z), y - z \rangle \leq \nu \|y - z\|^2$$

for some inner product. We consider the linear multistep method

$$\sum_{i=0}^{k} \alpha_i y_{m+i} = h \sum_{i=0}^{k} \beta_i f(x_{m+i}, y_{m+i})$$

(8.2)

together with its perturbed formula

$$\sum_{i=0}^{k} \alpha_i \tilde{y}_{m+i} = h \sum_{i=0}^{k} \beta_i f(x_{m+i}, \tilde{y}_{m+i}) + d_{m+k}.$$ 

(8.3)

The perturbations $d_{m+k}$ can be interpreted as the influence of round-off, as the error due to the iterative solution of the nonlinear equation, or as the local discretization error (compare Eq. (7.5)). Taking the difference of (8.3) and (8.2) we obtain (for $m \geq 0$)

$$\sum_{i=0}^{k} \alpha_i \Delta y_{m+i} = h \sum_{i=0}^{k} \beta_i \Delta f_{m+i} + d_{m+k},$$

(8.4)
where we have introduced the notation
\[ \Delta y_j = \tilde{y}_j - y_j, \quad \Delta f_j = f(x_j, \tilde{y}_j) - f(x_j, y_j). \] (8.5)

The one-sided Lipschitz condition cannot be used directly, because several \( \Delta f_j \) appear in (8.4) (in contrast to one-leg methods). In order to express one \( \Delta f_m \) in terms of \( \Delta y_j \) only we introduce the formal power series
\[ \Delta y(\zeta) = \sum_{j \geq 0} \Delta y_j \zeta^j, \quad \Delta f(\zeta) = \sum_{j \geq 0} \Delta f_j \zeta^j, \quad d(\zeta) = \sum_{j \geq 0} d_j \zeta^j. \]

It is convenient to assume that \( \Delta y_j = 0, \Delta f_j = 0, d_j = 0 \) for negative indices and that \( d_0, \ldots, d_{k-1} \) are defined by Eq. (8.4) with \( m \in \{-k, \ldots, -1\} \). Then Eq. (8.4) just compares the coefficient of \( \zeta^m \) in the identity
\[ \varrho(\zeta^{-1}) \Delta y(\zeta) = h \varrho(\zeta^{-1}) \Delta f(\zeta) + \zeta^{-k} d(\zeta). \] (8.4′)

Dividing (8.4′) by \( \sigma(\zeta^{-1}) \) and comparing the coefficients of \( \zeta^m \) yields
\[ \sum_{j=0}^{m} \delta_{m-j} \Delta y_j = h \Delta f_m + \tilde{d}_m, \] (8.6)

where
\[ \frac{\varrho(\zeta^{-1})}{\sigma(\zeta^{-1})} = \delta(\zeta) = \sum_{j \geq 0} \delta_j \zeta^j \] (8.7)
as in (7.45) and
\[ \frac{\zeta^{-k}}{\sigma(\zeta^{-1})} d(\zeta) = \tilde{d}(\zeta) = \sum_{j \geq 0} \tilde{d}_j \zeta^j. \] (8.8)

In (8.6) \( \Delta f_m \) is now isolated as desired and we can take the scalar product of (8.6) with \( \Delta y_m \). We then exploit the assumption (8.1) and obtain
\[ \sum_{j=0}^{m} \delta_{m-j} \text{Re} \langle \Delta y_j, \Delta y_m \rangle \leq h \nu \| \Delta y_m \|^2 + \text{Re} \langle \tilde{d}_m, \Delta y_m \rangle. \] (8.9)

This allows us to prove the following estimate.

**Lemma 8.1.** Let \( \{ \Delta y_j \} \) and \( \{ \Delta f_j \} \) satisfy (8.6) with \( \delta_j \) given by (8.7). If
\[ \text{Re} \langle \Delta f_m, \Delta y_m \rangle \leq \nu \| \Delta y_m \|^2, \quad m \geq 0, \]
and the method is \( A \)-stable, then there exist constants \( C \) and \( C_0 > 0 \) such that for\n\[ mh \leq x_{\text{end}} - x_0 \] and \( h \nu \leq C_0 \),
\[ \| \Delta y_m \| \leq C \sum_{j=0}^{m} \| \tilde{d}_j \|. \]