Chapter 11
Inequalities Related to Associative Variables

In this chapter, we introduce another class of dependent variables. Two r.v.’s $X$ and $Y$ are said to be positive quadrant dependent (PQD) if $P(X > x, Y > y) \geq P(X > x)P(Y > y)$ for any $x$, $y$; and negative quadrant dependent (NQD) if $P(X > x, Y > y) \leq P(X > x)P(Y > y)$.

A set of $n$ r.v.’s $X_1, \ldots, X_n$ is said to be (positive) associated (PA) if for any coordinatewise nondecreasing functions $f$ and $g$ on $R^n$, $\text{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$, whenever the covariance exists. The set is said to be negative associated (NA) if for any disjoint $A, B \subset \{1, \ldots, n\}$ and any nondecreasing functions $f$ on $R^A$ and $g$ on $R^B$, $\text{Cov}(f(X_k, k \in A), g(X_j, j \in B)) \leq 0$.

An infinite family of r.v.’s is said to be linearly positive quadrant dependent (LPQD) if for any disjoint integer sets $A$, $B$ and positive $a_j$’s, $\sum_{k \in A} a_k X_k$ and $\sum_{j \in B} a_j X_j$ are PQD; linearly negative quadrant dependent (LNQD) is obviously in an analogous manner. An infinite family of r.v.’s is said to be (positive) associated (resp. negative associated) if every finite subfamily is PA (resp. NA).

Clearly, for a pair of r.v.’s PQD (resp. NQD) is equivalent to PA (resp. NA). For a family of r.v.’s, PA (resp. NA) implies LPQD (resp. LNQD).

11.1 Covariance of PQD Variables

If $X$ and $Y$ are PQD (resp. NQD) r.v.’s, then

$$EXY \geq EXEY \quad \text{(resp. } EXY \leq EXEY\text{)},$$

whenever the expectations exist. The equality holds if and only if $X$ and $Y$ are independent.
Proof. Consider only the PQD case. If $F$ denotes the joint and $F_X$ and $F_Y$ the marginal distributions of $X$ and $Y$, then we have

$$EXY - EXEY = \int \int (F(x, y) - F_X(x)F_Y(y))dxdy,$$

(82)

which implies the desired inequality immediately from the definition of PQD.

Now suppose that the equality holds. Then $F(x, y) = F_X(x)F_Y(y)$ except possibly on a set of Lebesgue measure zero. From the fact that d.f.’s are right continuous, it is easily seen that if two d.f.’s agree almost everywhere with respect to the Lebesgue measure, they must agree everywhere. Thus $X$ and $Y$ are independent.


11.2 Probability of Quadrant on PA (NA) Sequence

Let $X_1, \ldots, X_n$ be PA (resp. NA) r.v.’s, $Y_j = f_j(X_1, \ldots, X_n)$ and $f_j$ be nondecreasing, $j = 1, \ldots, k$. Then for any $x_1, \ldots, x_k$,

$$P \left\{ \bigcap_{j=1}^k (Y_j \leq x_j) \right\} \geq \prod_{j=1}^k P\{Y_j \leq x_j\}$$

(resp. $P \left\{ \bigcap_{j=1}^k (Y_j \leq x_j) \right\} \leq \prod_{j=1}^k P\{Y_j \leq x_j\}$),

$$P \left\{ \bigcap_{j=1}^k (Y_j > x_j) \right\} \geq \prod_{j=1}^k P\{Y_j > x_j\}$$

(resp. $P \left\{ \bigcap_{j=1}^k (Y_j > x_j) \right\} \leq \prod_{j=1}^k P\{Y_j > x_j\}$).

Proof. Consider only the PA case. Clearly, $Y_1, \ldots, Y_k$ are PA. Let $A_j = \{Y_j > x_j\}$. Then $I_j = I(A_j)$ is nondecreasing in $Y_j$, and so, $I_1, \ldots, I_k$ are PA. Investigate increasing functions $f(t_1, \ldots, t_k) = t_1$ and $g(t_1, \ldots, t_k) = t_2 \cdots t_k$. $f(I_1, \ldots, I_k)$ and $g(I_1, \ldots, I_k)$ are PA and hence by 11.1,

$$E(I_1 I_2 \cdots I_k) \geq E(I_1)E(I_2 \cdots I_k).$$