\[\Omega\]-Arithmetization: A Discrete Multi-resolution Representation of Real Functions

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Abstract. Multi-resolution analysis and numerical precision problems are very important subjects in fields like image analysis or geometrical modeling. In the continuation of previous works of the authors, we expose in this article a new method called the \[\Omega\]-arithmetization. It is a process to obtain a multi-scale discretization of a continuous function that is a solution of a differential equation. The constructive properties of the underlying theory leads to algorithms which can be exactly translated into functional computer programs without uncontrolled numerical errors. An important part of this work is devoted to the definition and the study of the theoretical framework of the method. Some significant examples of applications are described with details.

Keywords: discrete geometry, nonstandard analysis, multi-resolution analysis, constructive mathematics.

1 Introduction

In some previous works \[68\], the authors have systematically studied a method of discretization called the arithmetization method. The principle of this method has led Reveillès to the definition of the discrete analytical line \[171819\]. This arithmetization process is a way to discretize a continuous curve solution of a differential equation. The informal point of view \[78\] that the real line \(\mathbb{R}\) is the same thing as the discrete line \(\mathbb{Z}\) seen from far away is the intuitive basis of this method. This could seem quite surprising since \(\mathbb{Z}\) is denumerable while \(\mathbb{R}\) is not but the theoretical framework we are working in is nonstandard analysis. Hence, denumerable sets are not exactly as in the “usual” framework The transformation from \(\mathbb{Z}\) to \(\mathbb{R}\) corresponds to a rescaling which induces a strong deformation of space. Moreover, in computer science, practical real numbers are only constructive ones that cannot be other than denumerable. So, from a practical point of view, our approach leads to nothing else than what we can expect

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in computer science. The arithmetization is obtained by transforming the usual integration Euler scheme used to compute the curves, solution to differential equations, into an equivalent integer scheme.

A rigorous implementation of this approach requires a model of the set $\mathbb{Z}$ of integer numbers together with a notion of infinitely large number (i.e. a scale on $\mathbb{Z}$). In the papers [6,3], such a model was introduced with the help of an axiomatic version of nonstandard analysis. The major drawback of this approach is that the infinitely large integers which arise in the corresponding method have only an axiomatic status. Consequently, in applications with concrete computations, it is impossible to give an exact numerical representation of these numbers; in such a situation, we are forced to choose sufficiently large values in an arbitrary manner. Hence, this choice is only a metaphoric representation of the theoretical framework.

In the present paper, we propose to rebuild the arithmetization method with the notion of $\Omega$-numbers introduced by Laugwitz and Schmieden [12][11][10]. Roughly speaking, an $\Omega$-number (natural, integer or rational) is a sequence of numbers of the same nature together with an adapted equality relation. The sets of $\Omega$-numbers are extending the corresponding sets of usual numbers with the added advantage of providing a natural concept of infinitely large integer numbers: for instance, an $\Omega$-integer $\alpha$ represented by a sequence $(\alpha_n)$ of integers is such that $\alpha \simeq +\infty$ if $\lim_{n \to +\infty} \alpha_n = +\infty$ in the usual meaning. Clearly, these infinite numerical entities are effectively constructive.

After having chosen an $\Omega$-integer $\omega$ such that $\omega \simeq +\infty$, we can define the Harthong-Reeb line $\mathcal{HR}_\omega$ [4] which is a numerical system consisting of $\Omega$-integers with the additional property of being roughly equivalent to the real line system. Not only the elements of $\mathcal{HR}_\omega$ have a constructive flavor, but we can show that the structure of this system partially fits with the constructive axiomatic developed by Bridges [1].

With this, it is possible to develop the $\Omega$-arithmetization as an arithmetization method based on this new framework. The principle of this method is unchanged and the resulting algorithm is formally the same. The new and crucial facts are the following:

- The algorithm operates on $\Omega$-numbers in a completely constructive way and consequently, in the applications, we can represent adequately all the entities present in the theory.
- The result of the algorithm appears to be an exact discrete multi-resolution representation of the continuous function on which the method is applied. See figure [1].

From the first point, we deduce that the implementation of the method does not lead to uncontrolled approximation errors. Even for the authors, the second point was a (good) surprise.

1 For instance, for the figure of [3] page 2024, we took $\beta = 50$ as an infinitely large number.