ON EIGENVALUES EIGENFUNCTIONS AND RESOLVENTS
OF GENERAL ELLIPTIC PROBLEMS

Shmuel Agmon

Introduction

In these lectures we shall describe some recent results concerning the spectral theory of general non-self-adjoint elliptic boundary value problems. We shall be interested in the following problems: (i) Completeness of eigenfunctions. (ii) Angular distribution of eigenvalues. (iii) Asymptotic distribution of eigenvalues. The general plan of the lectures is as follows. In Lecture I we shall introduce the general class of regular elliptic boundary value problems and discuss the growth of certain resolvents in the complex plane. In Lecture II we shall establish completeness results for eigenfunctions of general elliptic problems obtaining also some results on the angular distribution of eigenvalues. In Lecture III we shall discuss some special classes of elliptic problems such as self-adjoint problems and absolutely elliptic problems. In Lecture IV we shall describe a very general result on the asymptotic distribution of eigenvalues of non-self-disjoint elliptic problems.

We note that the first three lectures are taken from the author’s paper [1] which is due to appear shortly, whereas the material of the last lecture on the asymptotic distribution of eigenvalues is new.
Regular Elliptic Boundary Value Problems and Growth of Resolvents

We denote by \( G \) a bounded domain in \( n \)-space with boundary \( \partial G \) and closure \( \overline{G} \). We let \( x = (x_1, \ldots, x_n) \) be the generic point in \( E_n \) and use the notation:

\[
D_i = \frac{\partial}{\partial x_i}, \quad D = (D_1, \ldots, D_n),
\]
denoting by

\[
D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}
\]
a general derivative. Here \( \alpha \) stands for the multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) whose length \( \alpha_1 + \cdots + \alpha_n \) is denoted by \( |\alpha| \).

We consider complex valued functions \( u(x) \) defined in \( G \) (or \( \overline{G} \)). For \( u \in C^j(\overline{G}) \) we introduce the \( L^p \) norms (\( p \geq 1 \)):

\[
\| u \|_{L^p_j(G)} = \left( \sum_{|\alpha| \leq j} \int_G |D^\alpha u|^p \, dx \right)^{1/p}.
\]

The completion of \( C^j(\overline{G}) \) under the norm (1.1) is a Banach space of functions denoted here by \( H^j_{L^p_j}(G) \). If the boundary is Lipschitzian, \( H^j_{L^p_j}(G) \) coincides with the subclass of functions in \( L^p_j(G) \) whose derivatives in the distribution sense of order \( \ll j \) are functions belonging to \( L^p(G) \).

We shall denote by \( \mathcal{H}(x;D) \) an elliptic linear differential operator in \( \overline{G} \) (variable complex coefficients) of even order \( 2m \). Thus the characteristic