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ABOUT H-FUZZY DIFFERENTIATION

The concept of $H$-fuzzy differentiation is discussed thoroughly in the univariate and multivariate cases. Basic $H$-derivatives are calculated and then important theorems are presented on the topic, such as, the $H$-mean value theorem, the univariate and multivariate $H$-chain rules, and the interchange of the order of $H$-fuzzy differentiation. Finally is given a multivariate $H$-fuzzy Taylor formula. This treatment relies in [10].

2.1 Introduction

Fuzziness was first introduced in the celebrated paper [103]. For the notion of $H$-fuzzy derivative see [93] and [53]. First we give some background from Fuzziness, motivation and justification, necessary for the results to follow. In Propositions 2.5, 2.7, 2.8, 2.10 we calculate basic $H$-fuzzy derivatives. In Lemmas 1.13 and 1.14 we give results on fuzzy continuity, and in Propositions 2.13 and 2.14 we give basic properties of $H$-fuzzy differentiation. Then come the main results.

Theorem 2.15 is on $H$-Fuzzy Mean Value Theorem, Lemmas 2.16, 2.17 and 2.20 are auxiliary on fuzzy convergence and fuzzy continuity, Theorem 2.18 is on univariate $H$-fuzzy chain rule, and Theorem 2.19 is on multivariate $H$-fuzzy chain rule.

We conclude with Theorem 2.21 on the interchange of the order of $H$-fuzzy differentiation, and the development of a multivariate $H$-fuzzy Taylor formula with integral remainder, see Theorem 2.22 and Corollary 2.23.
2.2 Background

We need the Fuzzy Taylor formula

**Theorem 2.1** ([11], see also Theorem 15.14). Let \( T := [x_0, x_0 + \beta] \subset \mathbb{R} \), with \( \beta > 0 \). We assume that \( f^{(i)}: T \to \mathbb{R}_F \) are \( H \)-differentiable for all \( i = 0, 1, \ldots, n - 1 \), for any \( x \in T \). (I.e., there exist in \( \mathbb{R}_F \) the \( H \)-differences \( f^{(i)}(x + h) - f^{(i)}(x) \), \( f^{(i)}(x) - f^{(i)}(x - h) \), \( i = 0, 1, \ldots, n - 1 \) for all small \( h \): \( 0 < h < \beta \). Furthermore there exist \( f^{(i+1)}(x) \in \mathbb{R}_F \) such that the limits in \( D \)-distance exist and

\[
\lim_{h \to 0^+} \frac{f^{(i)}(x + h) - f^{(i)}(x)}{h} = \lim_{h \to 0^+} \frac{f^{(i)}(x) - f^{(i)}(x - h)}{h},
\]

for all \( i = 0, 1, \ldots, n - 1 \). Also we assume that \( f^{(n)} \), is fuzzy continuous on \( T \). Then for \( s \geq a; s, a \in T \) we obtain

\[
f(s) = f(a) \oplus f'(a) \odot (s - a) \oplus f''(a) \odot \frac{(s - a)^2}{2!} \oplus \cdots \oplus f^{(n-1)}(a) \odot \frac{(s - a)^{n-1}}{(n-1)!} \oplus R_n(a, s),
\]

where

\[
R_n(a, s) := (FR) \int_a^s \left( \int_a^{s_1} \cdots \left( \int_a^{s_{n-1}} f^{(n)}(s_n) ds_n \right) ds_{n-1} \right) \cdots ds_1.
\]

Here \( R_n(a, s) \) is fuzzy continuous on \( T \) as a function of \( s \).

**Note.** This formula is invalid when \( s < a \), as it is totally based on Corollary 1.12.

Next \( \mathcal{C}[0,1] \) stands for the class of all real-valued bounded functions \( f \) on \([0,1]\) such that \( f \) is left continuous for any \( x \in (0,1) \) and \( f \) has a right limit for any \( x \in [0,1) \), especially \( f \) is right continuous at \( 0 \). With the norm \( \|f\| = \sup_{x \in [0,1]} |f(x)| \), \( \mathcal{C}[0,1] \) is a Banach space [50].

We mention

**Theorem 2.2** (Wu and Ma [50]). For \( u \in \mathbb{R}_F \), denote \( j: j(u) := (u_-, u_+) \), where \( u_\pm = u_\pm(r) := u_\pm(r) \), \( 0 \leq r \leq 1 \). Then \( j(\mathbb{R}_F) \) is a closed convex cone with vertex 0 in \( \mathcal{C}[0,1] \times \mathcal{C}[0,1] \) (here \( \mathcal{C}[0,1] \times \mathcal{C}[0,1] \) is a Banach space with the norm defined by \( \|(f, g)\| := \max(\|f\|, \|g\|) \)), and \( j: \mathbb{R}_F \to \mathcal{C}[0,1] \times \mathcal{C}[0,1] \) satisfies

1. for all \( u, v \in \mathbb{R}_F \), \( s \geq 0, t \geq 0 \), \( j(su + tv) = sj(u) + tj(v) \),
2. \( D(u, v) = \|j(u) - j(v)\| \), i.e., \( j \) embeds \( \mathbb{R}_F \) into \( \mathcal{C}[0,1] \times \mathcal{C}[0,1] \) isometrically and isomorphically.