Chapter 2
Anosov and Circle Diffeomorphisms

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Abstract We present an infinite dimensional space of \(C^{1+}\) smooth conjugacy classes of circle diffeomorphisms that are \(C^{1+}\) fixed points of renormalization. We exhibit a one-to-one correspondence between these \(C^{1+}\) fixed points of renormalization and \(C^{1+}\) conjugacy classes of Anosov diffeomorphisms.

2.1 Introduction

The link between Anosov diffeomorphisms and diffeomorphisms of the circle is due to D. Sullivan and E. Ghys through the observation that the holonomies of Anosov diffeomorphisms give rise to \(C^{1+}\) circle diffeomorphisms that are \(C^{1+}\)
fixed points of renormalization (see also [2]). A. Pinto and D. Rand [23] proved that this observation gives one-to-one correspondence between the corresponding smooth conjugacy classes. After the works of Thurston [41] and Williams [43], a key object in this link is the smooth horocycle equipped with a hyperbolic Markov map.

2.2 Circle Diffeomorphisms

Fix a natural number $a \in \mathbb{N}$ and let $\mathcal{S}$ be a counterclockwise oriented circle homeomorphic to the circle $\mathbb{S}^1 = \mathbb{R}/(1 + \gamma)\mathbb{Z}$, where $\gamma = (-a + \sqrt{a^2 + 4})/2 = 1/(a + 1/(a + \cdots))$. We note that if $a = 1$ then $\gamma$ is the inverse of the golden number $(1 + \sqrt{5})/2$. A key feature of $\gamma$ is that it satisfies the relation $a\gamma + \gamma^2 = 1$.

An arc in $\mathcal{S}$ is the image of a non trivial interval $I$ in $\mathbb{R}$ by a homeomorphism $\alpha : I \to \mathcal{S}$. If $I$ is closed (resp. open) we say that $\alpha(I)$ is a closed (resp. open) arc in $\mathcal{S}$. We denote by $(a, b)$ (resp. $[a, b]$) the positively oriented open (resp. closed) arc in $\mathcal{S}$ starting at the point $a \in \mathcal{S}$ and ending at the point $b \in \mathcal{S}$. A $C^{1+}$ atlas $\mathcal{A}$ of $\mathcal{S}$ is a set of charts such that (i) every small arc of $\mathcal{S}$ is contained in the domain of some chart in $\mathcal{A}$, and (ii) the overlap maps are $C^{1+\alpha}$ compatible, for some $\alpha > 0$.

A $C^{1+}$ circle diffeomorphism is a triple $(g, \mathcal{S}, \mathcal{A})$ where $g : \mathcal{S} \to \mathcal{S}$ is a $C^{1+\alpha}$ diffeomorphism, with respect to the $C^{1+\alpha}$ atlas $\mathcal{A}$, for some $\alpha > 0$, and $g$ is quasi-symmetric conjugate to the rigid rotation $r_\gamma : \mathbb{S}^1 \to \mathbb{S}^1$, with rotation number equal to $\gamma/(1 + \gamma)$. We denote by $\mathcal{F}$ the set of all $C^{1+}$ circle diffeomorphisms $(g, \mathcal{S}, \mathcal{A})$, with respect to a $C^{1+}$ atlas $\mathcal{A}$ in $\mathcal{S}$.

In order to simplify the notation, we will denote the $C^{1+}$ circle diffeomorphism $(g, \mathcal{S}, \mathcal{A})$ only by $g$.

2.2.1 The Horocycle and Renormalization

Let us mark a point in $\mathcal{S}$ that we will denote by $0 \in \mathcal{S}$, from now on. Let $S_0 = [0, g(0)]$ be the oriented closed arc in $\mathcal{S}$, with endpoints 0 and $g(0)$. For $k = 0, \ldots, a$ let $S_k = [g^k(0), g^{k+1}(0)]$ be the oriented closed arc in $\mathcal{S}$, with endpoints $g^k(0)$ and $g^{k+1}(0)$ and such that $S_k \cap S_{k-1} = \{g^k(0)\}$. Let $S_{a+1} = [g^{a+1}(0), 0]$ be the oriented closed arc in $\mathcal{S}$, with endpoints $g^{a+1}(0)$ and 0. We introduce an equivalence relation $\sim$ in $\mathcal{S}$ by identifying the $a + 1$ points $g(0), \ldots, g^{a+1}(0)$ and form the topological space $H(\mathcal{S}, g) = \mathcal{S}/ \sim$ with the orientation induced by $\mathcal{A}$. We call this oriented topological space the horocycle (see Fig. 2.1) and we denote it by $H = H(\mathcal{S}, g)$. We consider the quotient topology in $H$. Let $\pi_g : \mathcal{S} \to H$ be the natural projection. The point $\xi = \pi_g(g(0)) = \cdots = \pi_g(g^{a+1}(0)) \in H$ is called the junction of the horocycle $H$. For every $k \in \{0, \ldots, a\}$, let $S_k^H = S_k^H(\mathcal{S}, g) \subset H$ be the projection by $\pi_g$ of the closed arc $S_k$. Let $R^H = S_0^H \cup S_{a+1}^H$ be the renormalized circle. The horocycle $H$ is the union of the renormalized circle $R^H$ with the circles $S_k^H$ for every $k \in \{1, \ldots, a\}$. A parametrization in $H$ is the image of a non trivial interval $I$ in $\mathbb{R}$ by a homeomorphism $\alpha : I \to H$. If $I$ is closed (resp.