2 Block-Structured Markov Chains

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Abstract In this chapter, the censoring technique is applied to be able to deal with any irreducible block-structured Markov chain, which is either discrete-time or continuous-time. The $R$, $U$- and $G$-measures are iteratively defined from two different censored directions: UL-type and LU-type. An important censoring invariance for the $R$- and $G$-measures is obtained. Using the censoring invariance, the Wiener-Hopf equations are derived, and then the UL- and UL-types of $RG$-factorizations are given. The stationary probability vector is given an $R$-measure expression; while the transient probability can be computed by means of the $R$, $U$- and $G$-measures. Finally, the $A$- and $B$-measures are proposed in order to discuss the state classification of the block-structured Markov chain.

Keywords stochastic model, block-structured Markov chain, the censoring technique, $R$-measure, $U$-measure, $G$-measure, $A$-measure, $B$-measure, censoring invariance, Wiener-Hopf equation, $RG$-factorization, state classification, stationary probability vector, transient probability, the first passage time.

In this chapter, the censoring technique is applied to be able to deal with any irreducible block-structured Markov chain with either finitely-many levels or infinitely-many levels, which is either discrete-time or continuous-time. Three probabilistic measures: $R$, $U$- and $G$-measures, are defined from two different censored directions: UL-type and LU-type, and an important censoring invariance for the $R$-and $G$-measures is obtained. Based on the censoring invariance, the Wiener-Hopf equations are derived, and the UL-and UL-types of $RG$-factorizations for the transition matrix are given. Furthermore, the $A$- and $B$-measures are proposed in order to discuss the state classification of the block-structured Markov chain. This chapter systemically develops the decomposition theory: the $RG$-factorizations, for any irreducible Markov chains. Based on the $RG$-factorizations, effective algorithms can be designed under a unified, constructive computational framework.
in order to deal with performance computation and system decision for practical stochastic models in many applied areas.

This chapter is organized as follows. Section 2.1 applies the censoring technique to deal with any irreducible discrete-time block-structured Markov chain. Some important properties of the censored Markov chain are given. For the discrete-time block-structured Markov chain, Sections 2.2 and 2.3 derive the UL- and LU-types of \( RG \)-factorizations, respectively. Section 2.4 provides an \( R \)-measure expression for the stationary probability vector of any positive recurrent block-structured Markov chain. Note that the \( R \)-measure expression is more effective than that in the literature. Specifically, those crucial formulae given in Neuts [26, 27] are simply re-derived by means of the \( R \)-measure expression. Section 2.5 defines \( A \)- and \( B \)-measures for any irreducible block-structured Markov chain, and constructs expressions for the \( A \)- and \( B \)-measures by means of the \( R \)-, \( U \)- and \( G \)-measures, respectively. Based on the \( A \)- and \( B \)-measures, necessary and sufficient conditions for the state classification of the block-structured Markov chain are obtained. Section 2.6 discusses the block-structured Markov chains with finitely-many levels. Section 2.7 gives the UL- and LU-types of \( RG \)-factorizations for any irreducible continuous-time block-structured Markov chain, and some useful results are summarized simply. Finally, Section 2.8 summarizes the references related to the results of this chapter.

### 2.1 The Censoring Chains

In this section, the censoring technique is applied to be able to deal with any irreducible discrete-time block-structured Markov chain. Also, some important properties for the censored Markov chains are given.

We consider an irreducible discrete-time block-structured Markov chain \( \{X_n, n \geq 0\} \) on the state space \( \Omega = \{(k, j) : k \geq 0, 1 \leq j \leq m_k\} \) whose transition probability matrix is given by

\[
P = \begin{pmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\
P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\
P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(2.1)

where \( P_{i,j} \) is a matrix of size \( m_i \times m_j \) whose \((r,r')\)th entry is given by

\[(P_{i,j})_{r,r'} = P\{X_{n+1} = (j,r') \mid X_n = (i,r)\}.
\]

In this chapter, we always assume that the Markov chain \( P \) is irreducible and stochastic (or substochastic). Note that the stochastics and substochastics are denoted as \( Pe = e \) and \( Pe \leq e \), respectively, where \( e \) is a column vector of ones with