Chapter 4
Constitutive Laws and Damage Theory

This chapter will discuss constitutive laws and basic invariant requirements in continuous media. To develop a continuous damage theory, the concepts of damage variables will be briefly introduced. The equivalent principles on continuum damage mechanics will be presented to obtain effective material properties. A large damage theory for anisotropic damaged materials will be discussed from the incremental complementary energy equivalence principle, and a few simple applications will be presented.

4.1. Constitutive equations

Constitutive laws of materials are developed from invariance requirements. The four basic invariance requirements are:

(i) The principle of determinism states “the behavior of materials at time $t$ is determined by all the past history of the motion of all material points in the body until time $t$”.

(ii) The principle of neighborhood states “the behavior of a material point $P$ at time $t$ is determined by the behavior of an arbitrary small neighborhood”.

(iii) The principle of coordinate invariance states “the constitutive laws of materials are independent of coordinates”.

(iv) The principle of material objectivity states “the constitutive laws of materials are independent of the rigid motion of the spatial coordinates”.

The detailed discussion of the objectivity of stress and strain tensors can be referred to Eringen (1962), Guo (1980) and Marsden and Hughes (1983).

To develop the constitutive laws, the following two assumptions are extensively adopted.

(A1) The natural states lie in the zero stress in the initial configuration $\mathcal{B}$. 
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(A2) The behaviors of materials are dependent on the current deformation state \( \mathbf{b} \) to the natural state.

From the foregoing hypothesis, there is a relation of the stress and strain, i.e.,

\[
\mathbf{t} = f(\mathbf{c}, \mathbf{P}).
\]  

(4.1)

Because \( \mathbf{c} = \mathbf{F} \cdot (\mathbf{F})^{T} = \mathbf{F} \cdot (\mathbf{F})^{T} \), the fundamental deformation theorem gives

\[
\mathbf{F} = \mathbf{c} \cdot \mathbf{R} \quad \text{and} \quad \mathbf{F} = \mathbf{R} \cdot \mathbf{C};
\]  

(4.2)

and also

\[
\mathbf{F} = \mathbf{c} \cdot \mathbf{R} \quad \text{and} \quad \mathbf{F} = \mathbf{R} \cdot \mathbf{C},
\]  

(4.3)

then,

\[
\mathbf{III}(\mathbf{F}) = \mathbf{III}(\mathbf{R} \cdot \mathbf{C}) = \mathbf{III}(\mathbf{R}) \cdot \mathbf{III}(\mathbf{C}).
\]  

(4.4)

Thus,

\[
\frac{\mathbf{III}(\mathbf{F})}{\sqrt{\mathbf{III}(\mathbf{C})}} = \frac{\mathbf{III}(\mathbf{F})}{\sqrt{\mathbf{III}(\mathbf{F} \cdot (\mathbf{F})^{T})}} = 1.
\]  

(4.5)

The orthogonal tensor \( \mathbf{R} \) represents the rotation only, which will be independent of deformation from the initial configuration \( \mathcal{B} \) to the deformed configuration \( \mathcal{B} \).

From \( \mathbf{F} = \mathbf{c} \cdot \mathbf{R} \cdot \mathbf{I} \) and \( \mathbf{F} = \mathbf{I} \cdot \mathbf{R} \cdot \mathbf{C} \),

\[
\mathbf{R} \cdot \mathbf{C} = \mathbf{c} \cdot \mathbf{R}.
\]  

(4.6)

With \( \mathbf{C} \cdot \mathbf{N} = \Lambda_r \mathbf{N} \), left multiplication of \( \mathbf{N} \) of the foregoing equation yields

\[
\mathbf{c} \cdot \mathbf{R} \cdot \mathbf{N} = \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{N} = \Lambda_r \mathbf{R} \cdot \mathbf{N}.
\]  

(4.7)

Using \( \mathbf{c} \cdot \mathbf{n} = \lambda_r \mathbf{n} \) and \( \lambda_r = \Lambda_r \),

\[
\mathbf{n} = \mathbf{R} \cdot \mathbf{N}.
\]  

(4.8)