Chapter 3
The $\ell_2$ Baire Sum

In this chapter we will present the notion of a Schauder tree basis and the construction of an $\ell_2$ Baire sum, both introduced in [AD].

Schauder tree bases will serve as technical devices for producing universal spaces for certain classes of Banach spaces. Their critical rôle will be revealed in Chap. 7.

To every Schauder tree basis we associate its $\ell_2$ Baire sum. It is a separable Banach space that contains, in a natural way, an isomorphic copy of every space in the class coded by the Schauder tree basis. The main goal achieved by this construction is that it provides us with an efficient “gluing” procedure. However there is a price we have to pay. Namely, the $\ell_2$ Baire sum contains subspaces which are “orthogonal” to all spaces in that class. Most of the material in this chapter is devoted to the study of these subspaces.

3.1 Schauder Tree Bases

Definition 3.1. [AD] Let $X$ be a Banach space, $\Lambda$ a countable set, $T$ a pruned B-tree on $\Lambda$, and $(x_t)_{t \in T}$ a normalized sequence in $X$ (with possible repetitions) indexed by the tree $T$. We say that $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t \in T})$ is a Schauder tree basis if the following are satisfied:

1. $X = \overline{\text{span}}\{x_t : t \in T\}$.
2. For every $\sigma \in [T]$ the sequence $(x_{\sigma|n})_{n \geq 1}$ is a bi-monotone basic sequence.

We recall that a B-tree $T$ on $\Lambda$ is a downwards closed subset of $\Lambda^{<\mathbb{N}}$ consisting of non-empty finite sequences (see Sect. 1.2); equivalently $T$ is a B-tree on $\Lambda$ if $T \cup \{\emptyset\}$ is a tree on $\Lambda$.

For every Schauder tree basis $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t \in T})$ and every $\sigma \in [T]$ we let

$$X_\sigma = \overline{\text{span}}\{x_{\sigma|n} : n \geq 1\}. \quad (3.1)$$
Notice that in Definition 3.1 we do not assume that the subspace $X_\sigma$ of $X$ is complemented. Also notice that if $\sigma, \tau \in [T]$ with $\sigma \neq \tau$, then this does not necessarily imply that $X_\sigma \neq X_\tau$. Let us give some examples of Schauder tree bases.

**Example 3.1.** Consider a Banach space $X$ with a normalized bi-monotone Schauder basis $(e_n)$. We set $\Lambda = \mathbb{N}$ and $T = \Sigma$, where by $\Sigma$ we denote the B-tree on $\mathbb{N}$ consisting of all non-empty finite strictly increasing sequences in $\mathbb{N}$ (see Sect. 1.2). Notice that for all $t \in \Sigma$ we have $|t| \geq 1$. We define $x_t = e_{|t|-1}$ for every $t \in \Sigma$. Then $X = (X, \mathbb{N}, \Sigma, (x_t)_{t \in \Sigma})$ is a Schauder tree basis. Observe that $X_\sigma = X$ for every $\sigma \in [\Sigma]$.

**Example 3.2.** As in Example 3.1 let $X$ be a Banach space with a normalized bi-monotone Schauder basis $(e_n)$. For every $t \in \Sigma$ let $m_t = \max\{n : n \in t\}$ and define $x_t = e_{m_t}$. Again we see that $\mathfrak{X} = (X, \mathbb{N}, \Sigma, (x_t)_{t \in \Sigma})$ is a Schauder tree basis. Notice that for every $\sigma \in [\Sigma]$ the sequence $(x_\sigma|_n)_{n \geq 1}$ is just a subsequence of $(e_n)$. Conversely, for every subsequence $(e_{t_n})$ of $(e_n)$ there exists a (unique) branch $\sigma \in [\Sigma]$ such that $(x_\sigma|_{t_n})_{n \geq 1}$ is the subsequence $(e_{t_n})$. That is, $\mathfrak{X} = (X, \mathbb{N}, \Sigma, (x_t)_{t \in \Sigma})$ is obtained by “spreading” all subsequences of $(e_n)$ along the branches of $\Sigma$.

### 3.2 The $\ell_2$ Baire Sum of a Schauder Tree Basis

**Definition 3.2.** [AD] Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t \in T})$ be a Schauder tree basis. The $\ell_2$ Baire sum of $\mathfrak{X}$, denoted by $T_2^\mathfrak{X}$, is defined to be the completion of $c_{00}(T)$ equipped with the norm

$$\|z\|_{T_2^\mathfrak{X}} = \sup \left\{ \left( \sum_{i=0}^l \left( \sum_{t \in s_i} z(t)x_t \right)^2 \right)^{1/2} \right\},$$

where the above supremum is taken over all finite families $(s_i)_{i=0}^l$ of pairwise incomparable segments of $T$.

Let $\mathfrak{X} = (X, \Lambda, T, (x_t)_{t \in T})$ be a Schauder tree basis. Let us gather some basic properties of the space $T_2^\mathfrak{X}$ associated to $\mathfrak{X}$.

**A.** We denote by $(e_t)_{t \in T}$ the standard Hamel basis of $c_{00}(T)$. We fix a bijection $h_T : T \to \mathbb{N}$ such that for every $t, s \in T$ with $t \sqsubset s$ we have $h_T(t) < h_T(s)$. We enumerate the tree $T$ as $(t_n)$ according to the bijection $h_T$. If $(e_{t_n})$ is the corresponding enumeration of $(e_t)_{t \in T}$, then the sequence $(e_{t_n})$ defines a normalized bi-monotone Schauder basis of $T_2^\mathfrak{X}$. For every $x \in T_2^\mathfrak{X}$ by $\text{supp}(x)$ we denote the support of $x$, that is, the set \{ $t \in T : x(t) \neq 0$ \}. The range of $x$, denoted by $\text{range}(x)$, is the minimal interval $I$ of $\mathbb{N}$ such that $\text{supp}(x) \subseteq \{ t_n : n \in I \}$. We isolate, for future use, the following simple fact.