Chapter 1
Fractional Zaslavsky and Hénon Discrete Maps

Vasily E. Tarasov

Abstract This paper is devoted to the memory of Professor George M. Zaslavsky passed away on November 25, 2008. In the field of discrete maps, George M. Zaslavsky introduced a dissipative standard map which is called now the Zaslavsky map. G. Zaslavsky initialized many fundamental concepts and ideas in the fractional dynamics and kinetics. In this chapter, starting from kicked damped equations with derivatives of non-integer orders we derive a fractional generalization of discrete maps. These fractional maps are generalizations of the Zaslavsky map and the Hénon map. The main property of the fractional differential equations and the correspondent fractional maps is a long-term memory and dissipation. The memory is realized by the fact that their present state evolution depends on all past states with special forms of weights.

1.1 Introduction

There are a number of distinct areas of mechanics and physics where the basic problems can be reduced to the study of simple discrete maps. Discrete maps have been used for the study of dynamical problems, possibly as a substitute of differential equations (Sagdeev et al., 1988; Zaslavsky, 2005; Chirikov, 1979; Schuster, 1988; Collet and Eckman, 1980). They lead to a much simpler formalism, which is particularly useful in computer simulations. In this chapter, we consider discrete maps that can be used to study the evolution described by fractional differential equations (Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006).

The treatment of nonlinear dynamics in terms of discrete maps is a very important step in understanding the qualitative behavior of continuous systems described by differential equations. The derivatives of non-integer orders (Samko et al., 1993) are
a natural generalization of the ordinary differentiation of integer order. Note that the continuous limit of discrete systems with power-law long-range interactions gives differential equations with derivatives of non-integer orders with respect to coordinates (Tarasov and Zaslavsky, 2006; Tarasov, 2006). Fractional differentiation with respect to time is characterized by long-term memory effects that correspond to intrinsic dissipative processes in the physical systems. The memory effects to discrete maps mean that their present state evolution depends on all past states. The discrete maps with memory are considered in the papers (Fulinski and Kleczkowski, 1987; Fick et al., 1991; Giona, 1991; Hartwich and Fick, 1993; Gallas, 1993; Stanislavsky, 2006; Tarasov and Zaslavsky, 2008; Tarasov, 2009; Edelman and Tarasov, 2009). The interesting question is a connection of fractional equations of motion and the discrete maps with memory. This derivation is realized for universal and standard maps in (Tarasov and Zaslavsky, 2008; Tarasov, 2009).

It is important to derive discrete maps with memory from equations of motion with fractional derivatives. It was shown (Zaslavsky et al., 2006) that perturbed by a periodic force, the nonlinear system with fractional derivative exhibits a new type of chaotic motion called the fractional chaotic attractor. The fractional discrete maps (Tarasov and Zaslavsky, 2008; Tarasov, 2009) can be used to study a new type of attractors that are called pseudochaotic (Zaslavsky et al., 2006).

In this chapter, fractional equations of motion for kicked systems with dissipation are considered. Correspondent discrete maps are derived. The fractional generalizations of the Zaslavsky map and the Hénon map are suggested. In Sect. 1.2, we give a brief review of fractional derivatives to fix notation and provide a convenient reference. In Sect. 1.3, the fractional generalizations of the Zaslavsky map are suggested. A brief review of well-known discrete maps is considered to fix notations and provide convenient references. In Sect. 1.4, the fractional generalizations of the Hénon map are considered. The differential equations with derivatives of non-integer orders with respect to time are used to derive generalizations of the discrete maps. In Sect. 1.5, a fractional generalization of differential equation in which we use a fractional derivative of the order $0 \leq \beta < 1$ in the kicked term, i.e. the term of a periodic sequence of delta-function type pulses (kicks). The other generalization is suggested in (Tarasov and Zaslavsky, 2008). The discrete map that corresponds to the suggested fractional equation of order $0 \leq \beta < 1$ is derived. This map can be considered as a generalization of universal map for the case $0 < \beta < 1$. In Sect. 1.6, a fractional generalization of differential equation for a kicked damped rotator is suggested. In this generalization, we use a fractional derivative in the kicked damped term, i.e. the term of a periodic sequence of delta-function type pulses (kicks). The other generalization is also suggested in (Tarasov and Zaslavsky, 2008). The discrete map that corresponds to the suggested fractional differential equation is derived. Finally, a short conclusion is given in Sect. 1.7.