A $(4 + \epsilon)$-Approximation for the Minimum-Weight Dominating Set Problem in Unit Disk Graphs

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Abstract. We present a $(4 + \epsilon)$-approximation algorithm for the problem of computing a minimum-weight dominating set in unit disk graphs, where $\epsilon$ is an arbitrarily small constant. The previous best known approximation ratio was $5 + \epsilon$. The main result of this paper is a 4-approximation algorithm for the problem restricted to constant-size areas. To obtain the $(4 + \epsilon)$-approximation algorithm for the unrestricted problem, we then follow the general framework from previous constant-factor approximations for the problem: We consider the problem in constant-size areas, and combine the solutions obtained by our 4-approximation algorithm for the restricted case to get a feasible solution for the whole problem. Using the shifting technique (selecting a best solution from several considered partitionings of the problem into constant-size areas) we obtain the claimed $(4 + \epsilon)$-approximation algorithm. By combining our algorithm with a known algorithm for node-weighted Steiner trees, we obtain a 7.875-approximation for the minimum-weight connected dominating set problem in unit disk graphs.

1 Introduction

A subset $D \subseteq V$ of the vertices of an undirected graph $G = (V, E)$ is called a dominating set if every vertex in $V$ is contained in $D$ or has a neighbor in $D$. A vertex in $D$ is called a dominator, and we say that a dominator dominates itself and all its neighbors. The minimum dominating set problem (MDS) is to compute a dominating set of smallest size. MDS belongs to the classical $NP$-hard optimization problems listed in the book of Garey and Johnson \cite{GJ79}. MDS for general graphs is equivalent to the set cover problem, and can thus be approximated within a factor of $O(\log n)$ for graphs with $n$ vertices using a greedy algorithm (see, e.g., \cite{HK06}), but no better unless all problems in $NP$ can be solved in $n^{O(\log \log n)}$ time \cite{BS10}. If every vertex of the input graph is associated with a weight, the minimum-weight dominating set problem (MWDS) is to compute a dominating set of minimum weight. Approximation ratio $O(\log n)$ can also be achieved for the weighted set cover problem and thus for MWDS \cite{GJ79}. The variants of the problems where the dominating set is asked to be connected in the input graph are called, in an obvious way, the minimum connected dominating set.
problem (MCDS) and the minimum-weight connected dominating set problem (MWCDS), respectively. The best known approximation ratio for MWCDS in general graphs is $O(\log n)$ as well \cite{8}.

We consider the problem of computing a minimum-weight (connected) dominating set in unit disk graphs. A unit disk graph is a graph where every vertex is associated with a disk of unit radius in the plane and there is an edge between two vertices of the graph if the two corresponding disks intersect. These problems are $NP$-hard already for the unweighted case \cite{4,12}. We are thus interested in approximation algorithms. An algorithm for MDS (or MWDS) is called a $\rho$-approximation algorithm, and has approximation ratio $\rho$, if it runs in polynomial time and always outputs a dominating set whose size (or total weight) is at most a factor of $\rho$ larger than the size (or total weight) of the optimal solution. The definitions for MCDS and MWCDS are analogous. A polynomial-time approximation scheme (PTAS) is a family of approximation algorithms with ratio $1 + \varepsilon$ for every constant $\varepsilon > 0$.

Constant-factor approximation algorithms for MDS and MCDS in unit disk graphs were given by Marathe et al. \cite{13}. For MDS in unit disk graphs, a PTAS was presented by Hunt et al. \cite{11}, based on the shifting strategy \cite{2,9}. These algorithms, however, do not extend to the weighted version. In particular, the PTAS is based on the fact that the optimal dominating set for unit disks in a $k \times k$ square has size at most $O(k^2)$ and can thus be found in polynomial time using complete enumeration if $k$ is a constant. In the weighted case, there is no such bound on the size of an optimal (or near-optimal) solution, as an optimal solution may consist of a large number of disks with tiny weight. For MCDS in unit disk graphs, a PTAS was presented in \cite{3}. For unit disk graphs with bounded density, asymptotic fully polynomial-time approximation schemes (with running time polynomial in $\frac{1}{\varepsilon}$ and in the size of the input, but achieving ratio $1 + \varepsilon$ only for large enough inputs) were presented for MDS and MCDS in \cite{15}.

The first constant-factor approximation algorithms for MWDS and MWCDS in unit disk graphs were given by Ambühl et al. \cite{1}, with approximation ratios 72 and 89, respectively. Huang et al. \cite{10} presented approximation algorithms with approximation ratio $6 + \varepsilon$ and $10 + \varepsilon$, respectively. Currently the best approximation algorithm for MWDS is due to Dai and Yu \cite{5}, with approximation ratio $5 + \varepsilon$. Zou et al. \cite{17} present an approximation algorithm with ratio $2.5\rho < 3.875$ for the node-weighted Steiner tree problem in unit disk graphs, where $\rho = 1 + \frac{\ln 3}{2}$ is the best known approximation ratio for the classical Steiner tree problem \cite{14}. This result can be used to connect a dominating set by adding nodes of weight at most $2.5\rho$ times the weight of an optimal connected dominating set, yielding the currently best approximation ratio of 8.875 for MWCDS.

Our Results. We present a $(4 + \varepsilon)$-approximation algorithm for MWDS in unit disk graphs. Our algorithm is based on several ideas of previous constant-factor approximation algorithms for the problem \cite{11,10}. We partition the plane into areas of size $K \times K$, where $K$ is an arbitrary constant. For each of these areas we consider the following subproblem: find a minimum-weight set of disks that dominate all disks that have a center in the area. The union of feasible solutions for each subproblem