Chapter 2
Adapting Mutational Equations to Examples in Vector Spaces: Local Parameters of Continuity

The notion of transitions instead of affine linear maps in a given direction has proved to be very powerful. Aubin’s definition of transition (Definition 1.1), however, is too restrictive.
Indeed, many examples in vector spaces share the feature that the Lipschitz constant of $t \mapsto \vartheta(t, x)$ cannot be bounded uniformly for all initial states $x$. In this chapter we will study several examples in which the transitions are based on solutions to linear problems in vector spaces. Doubling the initial state implies doubling the transition value and thus doubling the Lipschitz constant with respect to time.

The main goal of the subsequent chapters is to weaken the conditions on transitions and solutions in the mutational framework such that Euler method still provides existence of (generalized) solutions.
In this chapter, we implement two additional aspects in the recently introduced terms: Firstly, we use an analog of the absolute value in the metric space $(E, d)$. Indeed, $[\cdot] : E \to [0, \infty]$ is just to specify the “absolute magnitude” of each element in $E$, but does not have to satisfy structural conditions such as homogeneity or triangle inequality. In contrast to a metric, $[\cdot]$ does not serve the comparison of two elements in $E$, but the continuity parameters $\alpha(\vartheta), \beta(\vartheta)$ will be assumed to be uniform in all “balls” $\{x \in E \mid [x] \leq r\}$ with positive “radius” $r > 0$. The proofs do not change substantially if we impose appropriate bounds on the growth of $[\vartheta(\cdot, x)]$ for each initial element $x \in E$.

Secondly, we admit more than just one distance function on $E$ simultaneously. A family $(d_j)_{j \in J}$ of pseudo-metrics on $E$ (i.e. reflexive, symmetric and satisfying the triangle inequality, but not necessarily positive definite) replaces the metric $d$ always used in Chapter 1. The weak topology of a Banach space, for example, is much easier to describe by means of many linear forms than by just a single metric and, the suitable choice of linear forms will prove to be very helpful for semilinear evolution equations discussed in subsequent §2.4.

In a word, these extensions of the mutational framework do not require significant improvements of the proofs in comparison with the preceding chapter. They share the basic notion with later generalizations: For implementing additional “degrees of freedom”, we focus on the question which parameter may depend on which others.
2.1 The Topological Environment of This Chapter

$E$ always denotes a nonempty set, but we do not restrict our considerations to a metric space $(E, d)$ as in Chapter 1.

**Definition 1.** Let $E$ be a nonempty set. A function $d : E \times E \rightarrow [0, \infty]$ is called pseudo-metric on $E$ if it satisfies the following conditions:

1. $d$ is reflexive, i.e. for all $x \in E$: $d(x, x) = 0$,
2. $d$ is symmetric, i.e. for all $x, y \in E$: $d(x, y) = d(y, x)$
3. $d$ satisfies the triangle inequality, i.e. for all $x, y, z$: $d(x, z) \leq d(x, y) + d(y, z)$.

In particular, a pseudo-metric $d$ on $E$ does not have to be positive definite, i.e. $d(x, y) = 0$ does not always imply $x = y$.

**General assumptions for Chapter 2.** $E$ is a nonempty set and, $\mathcal{I} \neq \emptyset$ denotes an index set. For each index $j \in \mathcal{I}$, $d_j : E \times E \rightarrow [0, \infty]$ is a pseudo-metric on $E$ and, $[\cdot]_j : E \rightarrow [0, \infty]$ is a given function that is lower semicontinuous with respect to the topology of $(d_i)_{i \in \mathcal{I}}$, i.e. strictly speaking,

$$[x]_j \leq \liminf_{n \rightarrow \infty} |x_n|_j$$

for any $x \in E$ and sequence $(x_n)_{n \in \mathbb{N}}$ in $E$ with $d_i(x_n, x) \xrightarrow{n \rightarrow \infty} 0$ and $\sup_n |x_n|_i < \infty$ for each $i \in \mathcal{I}$.

Now the main goal of this chapter is to extend the mutational framework from a metric space to the tuple $(E, (d_j)_{j \in \mathcal{I}}, ([\cdot]_j)_{j \in \mathcal{I}})$. Several examples in vector spaces like semilinear evolution equations and nonlinear transport equations will follow.

2.2 Specifying Transitions and Mutation on $(E, (d_j)_{j \in \mathcal{I}}, ([\cdot]_j)_{j \in \mathcal{I}})$

**Definition 2.** $\vartheta : [0, 1] \times E \rightarrow E$ is called transition on $(E, (d_j)_{j \in \mathcal{I}}, ([\cdot]_j)_{j \in \mathcal{I}})$ if it satisfies the following conditions for each $j \in \mathcal{I}$:

1.) for every $x \in E$: $\vartheta(0, x) = x$
2.) for every $x \in E, t \in [0, 1]$: $\lim_{h \downarrow 0} \frac{1}{h} \cdot d_j(\vartheta(t + h, x), \vartheta(h, \vartheta(t, x))) = 0$
3.) there exists $\alpha_j(\vartheta; \cdot) : [0, \infty] \rightarrow [0, \infty]$, such that for any $x, y \in E$ with $[x]_j \leq r, [y]_j \leq r$:

$$\limsup_{h \downarrow 0} \frac{d_j(\vartheta(h, x), \vartheta(h, y))}{h} \leq \alpha_j(\vartheta; r) \cdot d_j(x, y)$$
4.) there exists $\beta_j(\vartheta; \cdot) : [0, \infty] \rightarrow [0, \infty]$, such that for any $s, t \in [0, 1]$ and $x \in E$ with $[x]_j \leq r$:

$$d_j(\vartheta(s, x), \vartheta(t, x)) \leq \beta_j(\vartheta; r) \cdot |t - s|$$
5.) there exists $\gamma_j(\vartheta) \in [0, \infty]$, such that for any $t \in [0, 1]$ and $x \in E$:

$$[\vartheta(t, x)]_j \leq ([x]_j + \gamma_j(\vartheta) \cdot \vartheta(t, x)) \cdot e^{\gamma_j(\vartheta)t}$$