Chapter 4

Search Procedure Exploiting Locally Regularized Objective Approximation: A Convergence Theorem for Direct Search Algorithms

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Abstract. The Search Procedure Exploiting Locally Regularized Objective Approximation is a method to speed-up local optimization processes in which the objective function evaluation is expensive. It was introduced in [1] and further developed in [2]. In this paper we present the convergence theorem of the method. The theorem is proved for the EXTREM [6] algorithm but applies to any Gauss-Siedle algorithm that uses sequential quadratic interpolation (SQI) as a line search method. After some extension it can also be applied to conjugate direction algorithms. The proof is based on the Zangwill theory of closed transformations. This method of the proof was chosen instead of sufficient decrease approach since the crucial element of the presented proof is an extension of the SQI convergence proof from [14] which is based on this approach.

4.1 Introduction

Optimization processes with objective functions that are expensive to evaluate – since usually their evaluation requires to solve a large system of linear equations or to simulate some physical process – occur in many fields of modern design. The main strategy in speeding up such processes via constructing a model to approximate an objective function are trust region methods [4]. The application of radial basis function approximation as an approximation model in trust region methods was discussed in [13]. The standard method to prove a convergence of a trust region method is the method of sufficient decrease.

In [1] and [2] we presented the search procedure which can be viewed as an alternative to trust region methods. It relies on combining the direct search algorithm EXTREM [6] with the locally regularized radial basis approximation. The
method achieves a speed-up of the optimization by exchanging some number of direct function evaluations by radial basis approximation. The method combined with the EXTREM algorithm was implemented within the computer program ROXIE for superconducting magnets design and optimization \cite{3}, however it is a general framework and as we shall see, any search algorithm that is based on the Gauss-Seidle non-gradient algorithm or conjugate direction algorithm, can be used. We will call our method the Search Procedure Exploiting Locally Regularized Objective Approximation (SPELROA).

In this paper we give the convergence theorem for the SPELROA method, combined with the EXTREM minimization algorithm under the assumption that the radial basis approximation has a relative error less than $\varepsilon$. The method of proof is different from those used in proofs of trust region methods since it is based on the Zangwill theory of closed transformations. The crucial element of the proof is the modification of the proof of the convergence of the quadratic interpolation as a line search method (c.f. \cite{14}) under the assumption that a perturbation of the function value may be introduced to the algorithm at each step. We give the conditions on the function values as well as on $\varepsilon$ to maintain convergence. The proof of the convergence of the sequential quadratic interpolation in \cite{14} is based on Zangwill’s theory and as we essentially extend it we chose this method also to prove the convergence of the whole SPELROA method.

The plan of this chapter is the following. In the next section we sketch the SPELROA method. In the third section we give some theory from \cite{21} to be used in the next section to prove the main result. Finally in the fourth section we discuss the radial basis approximation and heuristics used for its construction. We also describe difficulties in establishing strict error bound in the current state of the development of radial basis function approximation for sparse data. In the reminder of the paper we give numerical results for three test functions of 6, 8 and 11 variables from a set of test functions proposed in \cite{12}.

### 4.2 The Search Procedure

Let there be given a direct search optimization algorithm $A$ that uses the quadratic interpolation as a line search method. The SPELROA method combined with algorithm $A$ can be written in the form of the following algorithm (c.f. \cite{2}):

While generating the set $Z$ we have to take care that data points are not placed too close to each other. When two points are too close to each other – where a distance is controlled by a user-supplied parameter whose value is relative to the diameter of the set $Z$ – one of the points has to be replaced by another point not yet included. Such procedure of constructing $Z$ keeps the separation distance (c.f. \cite{16}) greater than the user-supplied parameter value and therefore guarantees that the radial basis function interpolation matrix is not singular. The crucial step of the scheme is point 3, containing a threefold check for whether the approximation $\tilde{f}(x_k)$ can be used in the algorithm $A$ instead of $f(x_k)$ evaluated directly. Conditions being checked in steps 3.a) and 3.b) are related to the radial basis approximation and will be discussed