Chapter 4
Nonsmooth Temporal Transformations (NSTT)

Abstract. In this chapter, different versions of nonsmooth argument substitutions, specifically - nonsmooth time, are introduced with proofs of the related identities. Basic rules for algebraic, differential and integral manipulations are described. In particular, final subsections show how to implement nonsmooth argument substitutions in the differential equations. These impose two principal features on the dynamical systems by generating specific algebraic structures and switching formulations to boundary-value problems. Notice that the transformation itself imply no constraints on dynamical systems and easily applies to both smooth and nonsmooth systems. Any further steps, however, should account for physical properties of the related systems. Indeed, linear coordinate transformations can significantly simplify the linear dynamic problems. Weakly nonlinear coordinate transformations often play the major role in the quasi-linear theory. Further, dynamical systems with discontinuities can be simplified by means of appropriate non-smooth transformations of variables.

4.1 Non-smooth Time Transformations

Major features induced by nonsmooth temporal substitutions can be briefly listed as follows:

- Introducing non-smooth temporal variables, in particular triangular sine wave, brings the coordinates into the algebra of hyperbolic numbers$^1$.
- Under appropriate conditions, differentiation or integration of the coordinates keeps the result within the same algebra and therefore eases the corresponding manipulations with the dynamic systems;
- Explicit time argument can be used together with the nonsmooth time in order to describe amplitude and/or frequency modulated processes.

$^1$ As mentioned in Introduction, - complex numbers $X + Ye$, where $e^2 = 1$; see details in the next subsections.
Notice that the transformation itself is a preliminary stage of analysis finalized by specific boundary value problems on standard intervals. Then, appropriate methods must be applied to the boundary value problems according to the related physical content.

4.1.1 Positive Time

To begin with a simple illustration of the non-smooth positive time, consider the series

\[
\ln \left( 2 \cosh \frac{t}{2} \right) = \frac{t}{2} + \exp(-t) - \frac{1}{2} \exp(-2t) + \frac{1}{3} \exp(-3t) - ... \tag{4.1}
\]

which is convergent for \( t \geq 0 \) but obviously becomes divergent if \( t < 0 \); the convergence is conditional at \( t = 0 \).

Nevertheless, replacing \( t \rightarrow |t| \), makes this series convergent for any \( t \) as follows

\[
\ln \left( 2 \cosh \frac{t}{2} \right) \equiv \ln \left( 2 \cosh \frac{|t|}{2} \right)
= \frac{|t|}{2} + \exp(-|t|) - \frac{1}{2} \exp(-2|t|) + \frac{1}{3} \exp(-3|t|) - ... \tag{4.2}
\]

An ‘side effect’ of such manipulation is that finite sums of the series are losing differentiability at \( t = 0 \), whereas the original function is smooth everywhere. However, it will be shown later that the series can be rearranged in such a manner that any truncated series becomes differentiable at \( t = 0 \) as many times as needed.

Note that the transformation \( t \rightarrow |t| \) is non-invertible. As a result, the manipulation illustrated by example (4.2) may not work for other cases. In the above example though, the substitution \( t \rightarrow |t| \) holds for both positive and negative time \( t \) due to evenness of the original function.

In order to extend the above idea on the general case, let us represent the time argument in the form

\[
t = |t| \sqrt{t} \tag{4.3}
\]

Now, taking into account the relationship \((|t|\sqrt{t})^2 = 1\), gives sequentially

\[
t^{2n} = |t|^{2n} \quad \text{and} \quad t^{2n+1} = (|t|)^{2n+1} |t|, \quad n = 1, 2, ...
\tag{4.4}
\]

Example 1. Combining identities (4.4) and the power series expansion of the exponential function, gives

\[
\exp(t) \equiv \exp(|t| \sqrt{t}) = 1 + \frac{|t|^2}{2!} + ... + \left( |t| + \frac{|t|^3}{3!} + ... \right) |t|.
\]