On Drawn K-In-A-Row Games

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Abstract. In 2005, Wu and Huang [9] presented a generalized family of k-in-a-row games. The current paper simplifies the family to Connect(\(k, p\)). Two players alternately place \(p\) stones on empty squares of an infinite board in each turn. The player who first obtains \(k\) consecutive stones of his own horizontally, vertically, diagonally wins. A Connect(\(k, p\)) game is drawn if both have no winning strategy. Given \(p\), this paper derives the value \(k_{\text{draw}}(p)\), such that Connect(\(k_{\text{draw}}(p), p\)) is drawn, as follows. (1) \(k_{\text{draw}}(2) = 11\). (2) For all \(p \geq 3\), \(k_{\text{draw}}(p) = 3p+3d+8\), where \(d\) is a logarithmic function of \(p\). So, the ratio \(k_{\text{draw}}(p)/p\) is approximate to 3 for sufficiently large \(p\). To our knowledge, our \(k_{\text{draw}}(p)\) are currently the smallest for all \(2 \leq p < 1000\), except for \(p = 3\).

1 Introduction

A generalized family of k-in-a-row games, called Connect(\(m, n, k, p, q\)), were introduced and presented by Wu et al. [9, 10]. Two players, named Black and White, alternately place \(p\) stones on empty squares of an \(mn\) board in each turn, except for that Black plays first and places \(q\) stones initially. The player who obtains \(k\) consecutive stones of his own first wins. Both players tie when the board is filled up without one winning. For example, Tic-tac-toe is Connect(3, 3, 3, 1, 1), Go-Moku in the free style (a traditional five-in-a-row game) is Connect(15, 15, 5, 1, 1), and Connect6 [10] played on the traditional Go board is Connect(19, 19, 6, 2, 1).

In the past, many researchers were engaged in understanding the theoretical values of Connect(\(m, n, k, p, q\)) games. Allis et al. [1, 2] solved Go-Moku with Black winning. Van den Herik et al. [6] and Wu et al. [9, 10] also mentioned several solved games for k-in-a-row games.

This paper is interested in drawn Connect(\(m, n, k, p, q\)) games, where both players have no winning strategy. More specifically, this paper only focuses on Connect(\(\infty, \infty, k, p, p\)) games, denoted by Connect(\(k, p\)) in this paper. Following strategy-stealing arguments raised by Nash (cf. [3]), Wu et al. [10] showed that White has no winning strategy. In order to prove whether games are drawn, we only need to show that Black has no winning strategy either. Given \(p\), this paper derives the value \(k_{\text{draw}}(p)\), such that

¹ For brevity, we use ‘he’ and ‘his’ whenever ‘he or she’ and ‘his or her’ are meant.
² Practically, stones are placed on empty intersections of Renju or Go boards. In this paper, when we say squares, we mean intersections.
Connect($k_{\text{draw}}(p)$, $p$) is drawn. Since a drawn Connect($k$, $p$) game also implies a drawn Connect($k+1$, $p$), the value $k_{\text{draw}}(p)$ should be as small as possible.

In the past, Zetters [11] derived that Connect(8, 1) is drawn. Pluhar derived tight bounds $k_{\text{draw}}(p) = p + O(\log_2 p)$ for all $p \geq 1000$ (cf. Theorem 1 in [8]). However, the requirement of $p \geq 1000$ is unrealistic in real games. Thus, it becomes important to obtain tight bounds when $p < 1000$. Recently, Hsieh and Tsai [7] derived that $k_{\text{draw}}(p) = 4p + 7$ for all positive $p$. So, the ratio $R = k_{\text{draw}}(p)/p$ is approximate to $4$.

In this paper, Theorem 1 (below) shows that $k_{\text{draw}}(2) = 11$, while the result in [7] is 15. Theorem 2 derives a general bound $k_{\text{draw}}(p) = 3p + 3d + 8$ for all $p \geq 3$, where $d$ is a logarithmic function of $p$, namely $P(d–1) < p \leq P(d)$ and $P(d) = 6 \times 2^d$–$d$–4. When compared with [7], our $k_{\text{draw}}(p)$ are smaller for all $p \geq 5$, and the same for $p = 4$. The ratio $R = k_{\text{draw}}(p)/p = 3 + (3d + 8)/p$ is approximated to $3$, which is smaller than $4$ in [7]. The proofs of both Theorem 1 and Theorem 2 are given in Section 2 and Section 3, respectively. Some open problems are given in Section 4.

**Theorem 1.** As described above, Connect(11, 2) is drawn.

**Theorem 2.** Consider all $p \geq 3$. Let $P(d–1) < p \leq P(d)$, where $P(d) = 6 \times 2^d$–$d$–4. Then, Connect$(3p + 3d + 8, p)$ are drawn.

### 2 Proof of Theorem 1

Before proving Theorem 1, we define a new game, called a **ConnectLine game**, as defined in Definition 1.

![Fig. 1. The game board $B_2$](image)

**Definition 1.** On a game board $B$ as in Connect($k$, $p$), a set of vertically, horizontally and diagonally straight lines are designated, marked as solid lines as illustrated in Fig. 1. Given such a game board $B$, the game ConnectLine($B$, $p$) is defined as follows.

1. The game rules are the same as Connect($k$, $p$), except for the following.
2. Black is allowed to place $p'$ stones on $B$, where $p' \leq p$. In next turn, White is allowed to place $p''$ stones, where $p'' \leq p'$.
3. Black wins when for some line all the squares of it are occupied by black stones.

The game ConnectLine($B$, $p$) is drawn if Black has no winning strategy, that is, White has some strategy such that Black cannot win in all cases.

The game boards described in Definition 1 can be viewed as **hypergraphs** [3, 5]. All squares are vertices, while all solid lines are so-called **hyperedges**. The goal of Black is to win by occupying all vertices of some hyperedge.

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Based on the strategy-stealing argument, White has no winning strategy.