REGULARITY LEMMAS FOR GRAPHS

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Szemerédi's regularity lemma proved to be a fundamental result in modern graph theory. It had a number of important applications and is a widely used tool in extremal combinatorics. For some further applications variants of the regularity lemma were considered. Here we discuss several of those variants and their relation to each other.

1. INTRODUCTION

Szemerédi's regularity lemma is one of the most important tools in extremal graph theory. It has many applications not only in graph theory, but also in combinatorial number theory, discrete geometry, and theoretical computer science. The first form of the lemma was invented by Szemerédi [47] as a tool for the resolution of a famous conjecture of Erdős and Turán [9] stating that any sequence of integers with positive upper density must contain arithmetic progressions of any finite length.

The regularity lemma roughly states that every graph may be approximated by a union of induced random-like (quasi-random) bipartite subgraphs. Since the quasi-randomness brings important additional information, the regularity lemma proved to be a useful tool. The regularity lemma allows one to import probabilistic intuition to deterministic problems. Moreover, there are many applications where the original problem did not suggest a probabilistic approach.

Motivated especially by questions from computer science, several other variants of Szemerédi's regularity lemma were considered. In Section 2

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we focus mainly on the lemmas proved by Frieze and Kannan [12] and by Alon, Fischer, Krivelevich, and M. Szegedy [2]. We show how these lemmas compare to Szemerédi's original lemma and how they relate to some other variants. Most proofs stated here appeared earlier in the literature and here we just give an overview. A thorough discussion of the connections of those regularity lemmas, from an analytical and geometrical perspective was given recently by Lovász and B. Szegedy in [30]. In Section 3 we discuss the so-called counting lemmas and the removal lemma and its generalizations. We close with a brief discussion of the limit approach of Lovász and B. Szegedy and its relation to the regularity lemmas from Section 2.

There are several surveys devoted to Szemerédi regularity lemma and its applications. The reader is recommended to consult Komlós and Simonovits [26] and Komlós, Shoukoufandeh, Simonovits, and Szemeredi [25], where many applications of the regularity lemma are discussed.

Another line of research, which we will not discuss here, concerns sparse versions of the regularity lemma. Since Szemerédi's lemma is mainly suited for addressing problems involving "dense" graphs, that is graphs with at least $\Omega(\binom{|V|}{2})$ edges, it is natural to ask for similar statements that would apply to "sparse graphs", i.e., graphs with $o(\binom{|V|}{2})$ edges. It turns out that a regularity lemma applicable to certain classes of sparse graphs can be proved [22, 34] (see also [1]). Such a lemma was first applied by Kohayakawa and his collaborators to address extremal and Ramsey-type problems for subgraphs of random graphs (see, e.g., [19, 20, 21]). Here we will not further discuss this line of research and we refer the interested reader to the surveys [15, 23, 31] and the references therein.

2. Regularity Lemmas

In this section we discuss several regularity lemmas for graphs. We start our discussion with the regularity lemma of Frieze and Kannan [12] in the next section. In Section 2.2 we show how Szemerédi's regularity lemma [48] can be deduced from the weaker lemma of Frieze and Kannan by iterated applications. In Section 2.3 we discuss the $(\varepsilon, r)$-regularity lemma, whose analog for 3-uniform hypergraphs was introduced by Frankl and Rödl [11]. We continue in Section 2.4 with the regularity lemma of Alon, Fischer, Krivelevich, and M. Szegedy [2], which can be viewed as an iterated version of Szemerédi's regularity lemma. In Section 2.5 we introduce the regular