Martingale Representations and Hedge Ratios

The calculation of hedge ratios is fundamental to both the valuation of derivative securities and also the risk management procedures needed to replicate these instruments. In Monte Carlo simulation the following results on martingale representations and hedge ratios will be highly relevant. In this chapter we follow closely Heath (1995) and consider the problem of finding explicit Itô integral representations of the payoff structure of derivative securities. If such a representation can be found, then the corresponding hedge ratio can be identified and numerically calculated. For simplicity, we focus here on the case without jumps. The case with jumps is very similar.

15.1 General Contingent Claim Pricing

In this section we introduce a general setting which expresses the price dynamics of a derivative security as the conditional expectation of the payoff structure of the security under a suitable pricing measure.

Financial Market Model

Let $W = \{W_t = (W_t^1, \ldots, W_t^m)^\top, t \geq 0\}$ be an $m$-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{A}_T, \mathbb{P}, P)$. We assume that the filtration $\mathcal{A} = (\mathcal{A}_t)_{t \in [0,T]}$ is the $P$-augmentation of the natural filtration of $W$, where we work with a bounded time horizon $T \in [0, \infty)$. These conditions ensure that $\mathcal{A}$ satisfies the usual conditions, see Sect. 1.2.

Let $X^{t_0,x} = \{X_t^{t_0,x} = (X_t^{1,t_0,x}, \ldots, X_t^{d,t_0,x})^\top, t \in [t_0, T]\}$ be a $d$-dimensional diffusion process that describes the factors in our financial market model and whose components satisfy the SDE

$$dX_t^{i,t_0,x} = a^i(t, X_t^{t_0,x}) \, dt + \sum_{j=1}^m b^{i,j}(t, X_t^{t_0,x}) \, dW_t^j \quad (15.1.1)$$

for \( t \in [t_0, T] \), \( i \in \{1, 2, \ldots, d\} \). Here \( X^{t_0, \mathbf{x}} \) starts at time \( t_0 \) with initial value \( \mathbf{x} = (x_1, \ldots, x_d)\top \in \mathbb{R}^d \). We assume that appropriate growth and Lipschitz conditions apply for the drift \( a^i \) and diffusion coefficients \( b^{i,j} \) so that (15.1.1) admits a unique strong solution and is Markovian, see Chap. 1.

In order to model the chosen numéraire in the market model we may select the first component \( X^{1,t_0,\mathbf{x}} = \beta = \{\beta_t, t \in [t_0, T]\} \) to model its price movements. Under real world pricing, \( \beta \) will be the growth optimal portfolio (GOP). However, under risk neutral pricing we use the savings account as numéraire. Both cases are covered by the following analysis.

The vector process \( X^{t_0,\mathbf{x}} \) could model several risky assets or factors that drive the securities, as well as, components that provide additional specifications or features of the model such as stochastic volatility, market activity, inflation or averages of risky assets for Asian options, etc.

### General Contingent Claims

In order to build a setting that will support American option pricing and certain exotic option valuations, as well as hedging, we consider a stopping time formulation as follows:

Let \( \Gamma_0 \subset [t_0, T] \times \mathbb{R}^d \) be some region with \( \Gamma_0 \cap (t_0, T] \times \mathbb{R}^d \) an open set and define a stopping time \( \tau : \Omega \rightarrow \mathbb{R} \) by

\[
\tau = \inf \{t > t_0 : (t, X^{t_0, \mathbf{x}}) \notin \Gamma_0 \}. \tag{15.1.2}
\]

Using the stopping time \( \tau \) we define the region

\[
\Gamma_1 = \left\{ (\tau(\omega), X^{t_0, \mathbf{x}}(\tau(\omega))) \in [t_0, T] \times \mathbb{R}^d : \omega \in \Omega \right\}.
\]

\( \Gamma_1 \) contains all points of the boundary of \( \Gamma_0 \) which can be reached by the process \( X^{t_0, \mathbf{x}} \). We now consider contingent claims with payoff structures of the form

\[ H_\tau = h(\tau, X^{t_0, \mathbf{x}}), \]

where \( h : \Gamma_1 \rightarrow \mathbb{R} \) is some payoff function.

Using a terminology that is applied often for American option pricing, we call the set \( \Gamma_0 \) the continuation region and \( \Gamma_1 \) the exercise boundary, which forms part of the stopping region. For a process \( X^{t_0, \mathbf{x}} \) with continuous sample paths, an option is considered ‘alive’ at time \( s \in [t_0, T] \), if \( (s, X^{t_0, \mathbf{x}}_s) \in \Gamma_0 \). On the other hand, it is ‘exercised’ or ‘stopped’ at the first time \( s \in [t_0, T] \) that \( (s, X^{t_0, \mathbf{x}}_s) \) touches the exercise boundary \( \Gamma_1 \). It is assumed that \( (t_0, \mathbf{x}) \in \Gamma_0 \), since otherwise the derivative security would be immediately ‘exercised’.

For example, if we take \( \Gamma_0 = [t_0, T] \times \mathbb{R}^d \), which implies \( \tau = T \) and payoff structures of the form \( h(T, X^{t_0, \mathbf{x}}_T) \), then this formulation reduces to the case of a multi-dimensional European style contingent claim. More generally, in the case of a process with jumps an option is ‘exercised’ or ‘stopped’ at the first time \( s \in [t_0, T] \) that \( (s, X^{t_0, \mathbf{x}}_s) \) leaves the continuation region.