Singularities of Hyperbolic Inhibited Shells

6.1 Introduction

This chapter is devoted to hyperbolic shells, whose principal curvatures are of opposite sign. Consequently, at each point of the middle surface, there are two asymptotic directions. Concerning the singularities emerging when \( \varepsilon \searrow 0 \), some aspects are very similar to the case of parabolic shells. For instance, singularities along characteristics are more singular than the loading \( f^3 \) (at least for the normal displacement \( u_3 \)) and propagate. However, the existence of two distinct families of characteristics implies important differences in the structures of these singularities with respect to parabolic shells. When the loading \( f^3 \) is singular along a characteristic, the resulting displacement \( u_3 \) is “only” two orders more singular than \( f^3 \), whereas it is four order more singular for parabolic shells (see section 2.6.2 of chapter 2). Moreover, some singular loadings (for instance a point force) propagate along both asymptotic directions (which is also different from the parabolic case).

An important consequence of the existence of two distinct families of characteristics is the apparition of a pseudo-reflection phenomenon which is described in section 2.7. When a singularity reaches a point of a boundary which is not a characteristic, it “reflects” along the two characteristics starting from this point of the boundary.

When considering a real problem, we may have propagation and reflection in the same time. It is then all more difficult to create “manually” an efficient mesh to describe these singularities, as in most of cases, there is no analytical solution predicting a priori the positions of the layers where the mesh needs to be refined. These complex situations, considered in section 6.4, will reveal once again all the interest and the necessity of an anisotropic adaptive mesh procedure to describe accurately the singularities.

6.2 The Limit Problem for a Hyperbolic Inhibited Shell

Taking a special parametrization \((y^1, y^2)\), where the coordinate lines correspond to the asymptotic curves of the middle surface, the covariant components \( b_{\alpha\beta} \)
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of the second fundamental form reduce to \( b_{11} = b_{22} = 0 \) and \( b_{12} \neq 0 \). In what follows, we still consider only a normal loading \( f^3 \) (with \( f^1 = f^2 = 0 \)), giving the most singular displacements. Thus, the membrane system (6.5) reduces to:

\[
\begin{align*}
D_1 T^{11} + D_2 T^{12} &= 0 \\
D_2 T^{22} + D_1 T^{12} &= 0 \\
-2b_{12} T^{12} &= f^3
\end{align*}
\] (6.1)

In the sequel, this system will be used to determine the higher order term of singularities of the three displacements (solutions of the limit problem) for various loadings. Numerical simulations using FE method and an adaptive anisotropic mesh procedure will confirm this result when \( \varepsilon \searrow 0 \) (see also [63, 40]).

6.2.1 Example of a Hyperbolic Paraboloid

In what follows, we consider the case of a shell whose middle surface \( S \) is a hyperbolic paraboloid (see Fig. 6.1). The surface \( S \) is defined by the mapping \( (\Omega, \Psi) \) with

\[
\Omega = \{(y^1, y^2) \in \mathbb{R}^2, (y^1, y^2) \in [-L, L]^2\}
\] (6.2)

and

\[
\Psi(y^1, y^2) = \left( y^1, y^2, \frac{y^1 y^2}{c} \right)
\] (6.3)

We will consider the values \( L = 50 \text{ mm} \) and \( c = 250 \text{ mm} \) for the numerical computations. Moreover, the material considered is isotropic and homogeneous, with a Young modulus \( E = 28,500 \text{ MPa} \) and a Poisson ratio \( \nu = 0.4 \).

The specific parametrization (6.3) corresponds to that of the asymptotic lines. Indeed, the mapping \( (\Omega, \Psi) \) implies the following covariant coefficients of the first and second fundamental forms:

\[
a_{\alpha\beta} = \begin{pmatrix}
1 + \frac{(y^2)^2}{c^2} & \frac{y^1 y^2}{c^2} \\
\frac{y^1 y^2}{c^2} & 1 + \frac{(y^1)^2}{c^2}
\end{pmatrix}
\] (6.4)

\[
b_{11} = b_{22} = 0 \quad b_{12} = b_{21} = \frac{1}{\sqrt{c^2 + (y^1)^2 + (y^2)^2}}
\] (6.5)

The only non-vanishing Christoffel symbols reduce to:

\[
\Gamma^1_{12} = \frac{y^2}{c^2 + (y^1)^2 + (y^2)^2}
\] (6.6)