Chapter 19
Fractional Dynamics of Hamiltonian Quantum Systems

19.1 Introduction

In the quantum mechanics, the observables are given by self-adjoint operators (Messiah, 1999; Tarasov, 2005). The dynamical description of a quantum system is given by a superoperator (Tarasov, 2008b), which is a rule that assigns to each operator exactly one operator. Dynamics of quantum observable is described by the Heisenberg equation. For Hamiltonian systems, the infinitesimal superoperator of the Heisenberg equation is defined by some form of derivation (Tarasov, 2005, 2008b). The infinitesimal generator \((i/\hbar)[H, \cdot]\), which is used in the Heisenberg equation, is a derivation of observables. A derivation is a linear map \(D\), which satisfies the Leibnitz rule \(D(AB) = (DA)B + A(DB)\) for all operators \(A\) and \(B\). Fractional derivative can be defined as a fractional power of derivative (see Section 5.7 in (Samko et al., 1993)). We consider a fractional derivative on a set of observables as a fractional power of derivative \((i/\hbar)[H, \cdot]\). It allows us to generalize a notion of quantum Hamiltonian systems. In this case, operator equation for quantum observables is a fractional generalization of the Heisenberg equation (Tarasov, 2008a). The suggested fractional Heisenberg equation is exactly solved for the Hamiltonians of free particle and harmonic oscillator. Fractional power of operators (Balakrishnan, 1960; Komatsu, 1966; Berens et al., 1968; Yosida, 1995; Krein, 1971; Martinez and Sanz, 2000) and superoperator (Tarasov, 2008b, 2009) can be used as a possible approach to describe an interaction between the system and an environment. We note that fractional power of the operator, which is defined by Poisson bracket of classical dynamics, was considered in (Tarasov, 2008c).

In Section 19.2, the fractional power of derivative and the fractional Heisenberg equation are suggested. In Section 19.3, the properties of time evolution, which is described by the fractional equation, are considered. In Section 19.4, the fractional dynamics of free particle is considered. In Section 19.5, we obtain a solution of fractional Heisenberg equation for linear harmonic oscillator. Finally, a short conclusion is given in Section 19.6.
19.2 Fractional power of derivative and Heisenberg equation

Quantum dynamics is described by superoperators (Tarasov, 2008b). A superoperator \( \mathcal{L} \) is a rule that assigns to each operator \( A \) exactly one operator \( \mathcal{L}(A) \). For Hamiltonian \( H \), let \( \mathcal{L}_H \) be a superoperator that is defined by the equation:

\[
\mathcal{L}_H A = \frac{1}{i\hbar}(HA - AH).
\]

Quantum dynamics of observables of Hamiltonian system is described by the operator differential equation:

\[
D_t A_t = -\mathcal{L}_H A_t.  \tag{19.1}
\]

Equation (19.1) is called the Heisenberg equation for Hamiltonian systems (Messiah, 1999; Tarasov, 2008b). The time evolution of a Hamiltonian system is induced by the Hamiltonian \( H \).

In order to obtain a fractional generalization of Eq. (19.1), we consider a concept of fractional power for \( \mathcal{L}_H \). If \( \mathcal{L}_H \) is a closed linear superoperator with an everywhere dense domain \( D(\mathcal{L}_H) \), having a resolvent \( R(z, \mathcal{L}_H) = (zI - \mathcal{L}_H)^{-1} \) on the negative half-axis, then there exists (Balakrishnan, 1960; Yosida, 1995; Krein, 1971) the superoperator:

\[
(\mathcal{L}_H)^\alpha \quad \text{for} \quad 0 < \alpha < 1.
\]

The superoperator \( (\mathcal{L}_H)^\alpha \) is defined on \( D(\mathcal{L}_H) \) for \( 0 < \alpha < 1 \). It is a fractional power of the Lie left superoperator. Using the superoperator (19.2), we can describe fractional dynamics of quantum systems by the equation:

\[
D_t A_t = -((\mathcal{L}_H)^\alpha A_t), \tag{19.3}
\]

where \( t \) and \( H/\hbar \) are dimensionless variables. Equation (19.3) is called the fractional Heisenberg equation.

**Remark.**

Equation (19.3) cannot be represented in the form:

\[
D_t A_t = -\mathcal{L}_{H_{\text{new}}} A_t = \frac{i}{\hbar}[H_{\text{new}}, A_t]
\]

with some operator \( H_{\text{new}} \). Therefore quantum systems described by (19.3) are not Hamiltonian systems. The systems will be called the fractional Hamiltonian systems. A set of usual Hamiltonian quantum systems is a special case of a set of fractional systems.

The Cauchy problem for operator equation (19.1) in which the initial condition is given at the time \( t = 0 \) by \( A_0 \), can be solved. Solution of this Cauchy problem can represented in the form \( A_t = \Phi_tA_0 \), where \( \Phi_t, t \geq 0 \) one-parameter superoperator.