Chapter 4
Electrodynamics of Fractal Distributions of Charges and Fields

4.1 Introduction

Joseph Liouville was a pioneer in development of fractional calculus to electrodynamics (Lutzen, 1985). The theory of fractional derivatives and integrals (Kilbas et al., 2006; Samko et al., 1993) can be applied to several specific electromagnetic problems (see, for example, (Engheta, 1997; Zelenyi and Milovanov, 2004; Milovanov, 2009; Potapov, 2005; Tarasov, 2008, 2009; Bogolyubov et al., 2009)). In this chapter, we consider electrodynamics of fractal distribution of charges and fields in the framework of fractional continuous model (Tarasov, 2005a,b, 2006a,b).

The linear, surface, or volume charge distributions of particles can be described by the amount of electric charge in a line, surface, or volume, respectively (Jackson, 1998; De Groot and Suttorp, 1972). In general, these distributions can be fractal, i.e., the charged particles form a set with non-integer-dimension. Therefore electric and magnetic fields of fractal distribution of charged particles and fields must be described. Fractal distribution can be described by fractional continuous model. In the general case, the fractal distribution of particles cannot be considered as a continuous distribution. There are points and domains that have no charges. In Refs. (Tarasov, 2005a,b, 2006a,b), we suggested to consider fractal distribution of charges and fields as a special continuous distribution. We use the procedure of replacement of the distribution with fractal dimension by some continuous model that uses fractional integrals. This procedure can be considered as a generalization of Christensen approach (Christensen, 2005), that leads us to use the fractional integration for fractal distributions. Using fractional continuous model for fractal distributions of charged particles and fields, we consider the electric and magnetic fields of these distributions (Tarasov, 2005a,b, 2006a,b).

In Sections 4.2–4.3, the densities of electric charge and current for fractal distribution are considered. In Sections 4.4–4.5, Gauss’ and Stokes’ theorems for fractal distributions in the framework of fractional continuous model are suggested. In Sections 4.6–4.9, we consider the simple examples of the fields of homogeneous fractal distribution. The Coulombs and Gauss’ laws, the Biot-Savart and Ampere's...
laws for fractal distribution in the framework of fractional continuous model are suggested. In Section 4.10, we consider the fractional generalization for integral Maxwell equation. In Section 4.11, we represent fractal distribution as an effective medium. In Sections 4.12–4.14, the examples of electric dipole and quadrupole moments for fractal distribution are considered. In Sections 4.15–4.16, the magnetohydrodynamic equations for fractal distribution of charged particles are discussed. Finally, a short conclusion is given in Section 4.17.

### 4.2 Electric charge of fractal distribution

The total electric charge that is distributed on the metric set $W$ with the dimension $D = 3$ with the density $\rho'(r',t)$ is defined by

$$Q_3(W) = \int_W \rho'(r',t) dV_3'$$

(4.1)

where

$$dV_3' = dx'dy'dz'$$

for Cartesian coordinates $x'$, $y'$, $z'$ with dimension $[x'] = [y'] = [z'] = \text{meter}$. We note that SI unit of $Q_3$ is Coulomb, and SI unit of $\rho'$ is Coulomb · meter$^{-3}$.

To generalize Eq. (4.1), we represent this equation through the dimensionless coordinate variables. We can introduce the dimensionless values:

$$x = x'/l_0, \quad y = y'/l_0, \quad z = z'/l_0, \quad r = r'/l_0,$$

where $l_0$ is a characteristic scale, and the charge density

$$\rho(r,t) = l_0^3 \rho'(r l_0,t),$$

where SI unit of $\rho$ is Coulomb, i.e., $|\rho| = \text{Coulomb}$. As a result, we obtain

$$Q_3(W) = \int_W \rho(r,t) dV_3,$$  

(4.2)

where $dV_3 = dx'dy'dz$ for dimensionless Cartesian coordinates. This representation allows us to generalize Eq. (4.2) to fractal distribution of charges.

In the fractional continuous model for fractal distribution of charges and fields, we use fractional integrals over a region of $\mathbb{R}^n$ instead of integrals over a fractal set. In order to describe fractal distribution by fractional continuous model, we use the notion of density of states $c_n(D,r)$ and the density of charges $\rho(r,t)$. The function $c_n(D,r)$ is a density of states in the $n$-dimensional Euclidean space $\mathbb{R}^n$. The density of states describes how permitted states are closely packed in the space $\mathbb{R}^n$. The density of charges $\rho(r,t)$ describes a distribution of charges on a set of permitted states in the Euclidean space $\mathbb{R}^n$. In the fractional continuous model of fractal media,