Fast Coupled Path Planning: From Pseudo-Polynomial to Polynomial* 

Yunlong Liu¹ and Xiaodong Wu¹,²

¹ Department of Electrical and Computer Engineering, the University of Iowa
² Department of Radiation Oncology, The University of Iowa

Abstract. The coupled path planning (CPP) problem models the motion paths of the leaves of a multileaf collimator for optimally reproducing the prescribed dose in intensity-modulated radiation therapy (IMRT). Two versions of the CPP problem, unconstrained and constrained CPPs, are studied based on whether specifying the starting and ending positions of the leaf paths. By exploring the underlying properties of the problem such as submodularity and \( L^2 \)-convexity, we solve both CPP problems in polynomial time using the path-end hopping, local searching and proximity scaling techniques, improving current best known pseudo-polynomial time algorithms. Our algorithms are simple and easy to be implemented. Experimental results on real medical data showed that our CPP algorithms outperformed previous best-known algorithm by at least one order of magnitude.

Keywords: Coupled path planning, IMRT, submodularity, \( L^2 \)-convexity.

1 Introduction

In this paper, we study an interesting geometric optimization problem, called coupled path planning (CPP). Given a non-negative integer reference function \( f \) defined on the integer set \( \{1, 2, \ldots, n\} \), and three positive integer parameters \( H \), \( c \) and \( \Delta \), the problem is defined on a uniform grid \( R_g \) of size \( n \times H \) such that the length of each grid edge is one unit. The goal is to seek two paths \( \mathbf{x}_l \) and \( \mathbf{x}_r \) in \( R_g \), called the left and right paths, which are (1) \textit{xy-monotone}: monotone with respect to both the \( x \)-axis and the \( y \)-axis; (2) \textit{c-steep}: every vertical segment of the paths is of length at least \( c \) and every horizontal segment is of unit length; (3) \textit{non-crossing}: two paths \( \mathbf{x}_l \) and \( \mathbf{x}_r \) do not get across each other; (4) \textit{\( \Delta \)-error}: the vertical distance between \( \mathbf{x}_l \) and \( \mathbf{x}_r \) at each column \( i \) approximates \( f(i) \) within the error bound \( \Delta \) (i.e., \(|\mathbf{x}_l(i) - \mathbf{x}_r(i) - f(i)| \leq \Delta \); and (5) \textit{optimized}: the total sum of the errors (i.e., \( \sum_{i=1}^{n} |\mathbf{x}_l(i) - \mathbf{x}_r(i) - f(i)| \)) is minimized.

We consider two versions of the CPP problem: (1) The \textit{unconstrained CPP} (UCPP), where the starting and ending positions of the sought paths are not given as part of the input; (2) the \textit{constrained CPP} (CCPP), in which the starting and ending positions of the sought paths are fixed.

* This research was supported in part by the NSF grants CCF-0830402 and CCF-0844765, and the NIH grant K25-CA123112.
Application Background. The CPP problems arise in a modern cancel therapy technique named Intensity-Modulated Radiation Therapy (IMRT). To use this technique effectively, the prescribed radiation dose distributions, namely intensity maps (IMs), should be delivered accurately and efficiently. An IM is normally specified by a set of nonnegative integers on a uniform 2D grid. The number in a grid cell indicates the amount of radiation to be delivered to the corresponding body region.

Multileaf collimator (MLC) \[16\] is one of the most advanced tools for IM delivery in IMRT. An MLC consists of a number of pairs of tungsten alloy leaves of the same rectangular shape and size. The leaves can move left and right (or up and down) to form a rectilinear region, called an MLC-aperture. The cross-section of a radiation beam is shaped by an MLC-aperture.

Two radiation delivery methods are commonly used in IMRT \[16\], called static and dynamic leaf sequencing. In both leaf sequencing methods, the key problem is to determine a delivery plan for a given IM, i.e. a description of how to position the MLC leaves and maintain the on/off states of the radiation to deliver the IM. Two key criteria among others are used to measure the quality of an IMRT delivery plan: (1) Delivery time (the efficiency): The reduction of delivery time is beneficial in reducing the intra-fraction uncertainties that occur with organ motions and in minimizing the loss of biological effectiveness \[8\], thus improving the efficacy of IMRT. (2) Delivery error (the accuracy): Various factors may cause the discrepancy between the prescribed radiation dose and the actually delivered dose.

As in Ref. \[5\], the unconstrained CPP problem models the delivery method of dynamic leaf sequencing. Note that an IM may need to be delivered with a number of MLC leaf pairs, with each pair delivering one row of the IM. Some MLCs (e.g., the Varian MLCs) have the capacity of delivering each IM row independently with one leaf pair. Thus, it suffices to consider an optimal way to deliver one IM row. In the CPP problem, \( f(\cdot) \) is the intensity profile (reference function) specifying one row of a given IM, and the two output paths, \( x_l \) and \( x_r \), are the moving trajectories of the two leaf-ends of the MLC leaf pair, i.e., the leaf-end positions (the \( x \)-coordinates) at any unit time of the delivery (the \( y \)-coordinates). Each MLC leaf has a maximum moving velocity. It takes at least \( c \) units of time (i.e., \( c \) vertical edges in \( R_g \)) for one leaf-end moves forward one unit length (i.e., one horizontal edge in \( R_g \)). Thus, the paths have to be \( c \)-steep for some \( c > 0 \). The vertical distance between two paths, \( x_l \) and \( x_r \), on the \( i \)-th column of \( R_g \) is the delivery time duration (the delivered intensity level) that the \( i \)-th entry of the IM row is exposed to irradiation. To ensure the treatment quality, the delivered intensity level should be close enough to the prescribed intensity level (say, within an error range of \( \Delta \)). The total error over all columns of \( R_g \) gives the delivery error incurred to the IM row specified by \( f(\cdot) \). Hence, the CPP problem seeks to deliver one IM row in \( H \) units of delivery time while minimizing the total delivery error.

The constrained CPP problem (CCPP) is a key to a newly emerging IMRT delivery technique called arc-modulated radiation therapy (AMRT) \[17\].