Chapter 9

Theory of elasticity of three-dimensional quasicrystals and its applications

In Chapters 5∼8 we discussed the theories of elasticity of one- and two-dimensional quasicrystals and their applications. In this chapter the theory and applications of elasticity of three-dimensional quasicrystals will be dealt with. The three-dimensional quasicrystals include icosahedral quasicrystals and cubic quasicrystals. In all about 200 individual quasicrystals observed to date there are almost 100 icosahedral quasicrystals, so that they play the central role in this kind of solids. This suggests the major importance of elasticity of icosahedral quasicrystals in the study of mechanical behaviour of quasicrystalline material.

There are some polyhedrons with the icosahedral symmetry, one among them is shown in Fig. 9.0-1, which consists of 20 right triangles and contains 12 five-fold symmetric axes $A_5$, 20 three-fold symmetric axes $A_3$ and 30 two-fold symmetric axes $A_2$. One of the diffraction pattern is shown in Fig. 3.1-1 and the stereographic structure of one of icosahedral point groups is also depicted in Fig. 3.1-1.

The elasticity of icosahedral quasicrystals was studied immediately after the discovery of the structure, which is the pioneering work of the field. The outlook about this was figured out in the Chapter 4, in which the contribution of pioneers such as P. Bak etc was introduced. Afterward Ding et al$^{[1]}$ set up the physical framework of elasticity of icosahedral quasicrystals, they$^{[2]}$ also summarized the basic relationship of elasticity of cubic quasicrystals. In terms of the Green function method, Yang et al$^{[3]}$ gave an approximate solution on dislocation for a special case, i.e., the phonon-phason decoupled plane elasticity of icosahedral quasicrystal. In this chapter we mainly discuss the general theory of elasticity of icosahedral quasicrystals and the applications, in addition, those for cubic quasicrystals are also concerned. We focus on the
mathematical theory of the elasticity and the analytic solutions. Because of the large number of field variables and field equations involving elasticity of these two kinds of three-dimensional quasicrystals, the solution presents tremendous difficulty. We continue to develop the decomposition procedure adopted in the previous chapters, this can reduce the number of the field variables and field equations, and three-dimensional elasticity can be simplified to two-dimensional elasticity to solve for some cases with important practical applications. The introducing of displacement potentials or stress potentials\cite{4,5} can further simplify the problems. In the work some systematic and direct methods of mathematical physics and function theory have been developed, and a series of analytic solutions are constructed, which will be included in the chapter. Because the calculations are very complex, we would like to introduce them in detail as much as possible in order to facilitate comprehension of the text.

9.1 Basic equations of elasticity of icosahedral quasicrystals

The equations of deformation geometry are

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad w_{ij} = \frac{\partial w_i}{\partial x_j}, \quad (9.1-1)$$

which are similar in form to those given in previous chapters, but here $u_i$ and $w_i$ have 6 components, and $\varepsilon_{ij}$ and $w_{ij}$ have 15 components in total.

The equilibrium equations are as follows:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial H_{ij}}{\partial x_j} = 0, \quad (9.1-2)$$

which are also similar in form to those listed in previous chapters, however here adding $\sigma_{ij}$ and $H_{ij}$ gives 15 stress components.

Between the stresses and strains there is the generalized Hooke’s law such as

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} + R_{ijkl} w_{kl}, \quad H_{ij} = R_{kl} \varepsilon_{kl} + K_{ijkl} w_{kl}, \quad (9.1-3)$$

in which the phonon elastic constants are described by

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (9.1-4)$$

where $\lambda$ and $\mu$ (=$G$ in some references) the Lamé constants.

If the strain components are arranged as a vector according to the order

$$[\varepsilon_{ij}, w_{ij}] = [\varepsilon_{11} \varepsilon_{22} \varepsilon_{33} \varepsilon_{23} \varepsilon_{31} \varepsilon_{12} w_{11} w_{22} w_{33} w_{23} w_{32} w_{12} w_{32} w_{13} w_{21}] \quad (9.1-5')$$