On Concordance Measures and Copulas with Fractal Support

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Abstract. Copulas can be used to describe multivariate dependence structures. We explore the rôle of copulas with fractal support in the study of association measures.

1 General Introduction and Motivation

Copulas are of interest because they link joint distributions to their marginal distributions. Sklar [12] showed that, for any real-valued random variables $X_1$ and $X_2$ with joint distribution $H$, there exists a copula $C$ such that $H(u,v) = C(F_1(u), F_2(v))$, where $F_1$ and $F_2$ denote the cumulative (or margin) distributions of $X_1$ and $X_2$, respectively. If the marginals are continuous, then the copula is unique. Notice that it is also true the converse implication of Sklar’s Theorem. In fact, we may link any univariate distributions with any copula in order to obtain a valid joint distribution function. An implication of Sklar Theorem is that the dependence among $X_1$ and $X_2$ is fully described by the associated copula. Indeed, most conventional dependence measures can be explicitly expressed in terms of the copula, and they are designed to capture certain aspects of dependence or association between random variables.

On the other hand, all the examples of singular copulas we have found in the literature are supported by sets with Hausdorff dimension 1. However, it is implicit in some papers, for example in [11], that the well-known examples of Peano and Hilbert curves provide self-similar copulas with fractal support, since the Hausdorff dimension of their graphs is 3/2.

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Recently, Fredricks et al. [7], using an iterated function system, constructed families of copulas whose supports are fractals. In particular, they give sufficient conditions for the support of a self-similar copula to be a fractal whose Hausdorff dimension is between 1 and 2.

In [1], the authors prove that a necessary and sufficient condition for a copula to be the independence (or product) copula $\Pi$ is that the pair of measure preserving transformations representing the copula be independent as random variables; and a general constructive method for representation of copulas in terms of measure preserving transformations is given. Specifically, we study the copulas introduced by Fredricks et al. in [7], through two representation number systems we construct *ad hoc*. This paper is devoted to study these copulas in depth.

Firstly, we give an example of copula with a support with a Lebesgue measure 0, and a Hausdorff dimension 2. Moreover, we study the coefficients of upper and lower tail dependence of these copulas. Finally, we explore some well-known measures of dependence, namely Kendall’s $\tau$, Spearman’s $\rho$, and Gini’s index $\gamma$. The results we prove coincide with those regard to the independence copula $\Pi$.

## 2 Preliminaries about Copulas with Fractal Support

This section contains background information and useful notation.

(1) Let $I$ be the closed unit interval $[0, 1]$ and let $I^2 = I \times I$ be the unit square. For an introduction to copulas see, for example, [4] or [9].

(2) A *transformation matrix* is a matrix $T$ with nonnegative entries, for which the sum of the entries is 1 and none row or column has zero as entry everywhere.

Following [7], we recall that each transformation matrix $T$ determines a subdivision of $I^2$ into subrectangles $R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j]$, where $p_i$ (respect. $q_j$) denotes the sum of the entries in the first $i$ columns (respect. $j$ rows) of $T$. For a transformation matrix $T$ and a copula $C$, $T(C)$ denotes the copula that, for each $(i, j)$, spreads mass on $R_{ij}$ in the same way in which $C$ spreads mass on $I^2$.

Theorem 2 in [7] shows that for each transformation matrix $T \neq [1]$, there is an unique copula $C_T$ such that $T(C_T) = C_T$.

(3) Let $T$ be a transformation matrix, and let us consider the following conditions:

(i) $T$ has at least one zero entry.

(ii) For each non-zero entry of $T$, the row and column sums through that entry are equal.

(iii) There is at least one row or column of $T$ with two nonzero entries.

Theorem 3 in [7] shows that if $T$ is a transformation matrix satisfying condition (i), then $C_T$ is singular (that is, its support has Lebesgue measure