Chapter 1
Some examples of functions of very many variables in natural science and technology

1.1 Digital sampling of a signal (CIM — code-impulse modulation)

1.1.1 The linear device and its mathematical description (convolution)

A good mathematical model of many devices and instruments is the linear operator. The term “linear device with time-invariant properties” in mathematical language means that it is a linear operator $A$ acting on functions of time and commuting with the shift operator $T$, that is, $AT = TA$, where $T = T_\tau$, and $(T_\tau f)(t) = f(t - \tau)$.

For example, if $A$ is a record player, then tomorrow it should produce from the same disc the same music as today, only with the natural shift in time. In accordance with the accepted radio-engineering terminology, the function $f$ on which the operator acts is called the signal, more precisely, the input signal or input, while the result $A f$ is called the signal at the output or output and is denote by $\tilde{f}$.

Since a continuous function $f$ can be well approximated by a step function, it is easy to see that if one knows the response of such a device $A$ to an elementary step-like datum then one can find its response to any input signal $f$.

Ideally the step-like datum is converted to the unit impulse, the $\delta$-function. If we write down the identity $f(t) = \int f(\tau) \delta(t - \tau) \, d\tau$, then we immediately see that $Af(t) = \int f(\tau) A \delta(t - \tau) \, d\tau = \int f(\tau) \delta(t - \tau) \, d\tau =: f * \delta$, where $*$ is the symbol for the operation of convolution of functions. Hence $Af = f * \delta$.

The function $A\delta = \tilde{\delta}$, that is, the response of the device to the unit impulse ($\delta$-function) is called the instrumental function of the device and is often denoted by the symbol $E$. Thus, from a mathematical point of view the device $A$ is simply the convolution operator $Af = f * \tilde{\delta} = f * E$. Thus the solution of convolution equations has very specific direct applications (for example, the recovery of a transmitted signal from the received signal $A f = \tilde{f}$).
1.1.2 Fourier-reciprocal (spectral) description of a linear device

We recall that the frequency $\nu$ of a periodic process is usually measured by the number of complete cycles in unit time (one Hertz is one complete oscillation in one second; it is denoted by 1Hz). The angular or circular frequency $\omega = 2\pi \nu$ differs from the frequency $\nu$ merely by the factor $2\pi$ corresponding to measurement in radians per unit time.

We shall calculate the response $\tilde{f}$ of the device to an input signal $f = e^{i\omega t}$ (hence, by Euler’s formula $e^{i\omega t} = \cos \omega t + i \sin \omega t$ we also know the response of the device to simple harmonic oscillation $\sin \omega t$ of angular frequency $\omega$; here it is convenient to use complex language):

$$Af = f * \delta = f * E = \int f(t - \tau)\delta(\tau)d\tau = \int e^{i\omega(t-\tau)}E(\tau)d\tau = \left(\int E(\tau)e^{-i\omega\tau}d\tau\right)e^{i\omega t} = P(\omega)e^{i\omega t}.$$

We have obtained an oscillation with the same frequency as the input, but, possibly, a change in amplitude by a factor $|P(\omega)|$ and a change of phase corresponding to $\arg P(\omega)$. The quantity $P$ as a function of $\omega$ is called the spectral characteristic of the device. It is clear that the spectral characteristic of the device is (to within a normalizing factor) the Fourier transform $\hat{E}$ of the instrumental function $E$ of this device: $P = 2\pi \hat{E}$. We stipulate that $p(\nu) := P(2\pi\nu) = P(\omega)$.

Recall that in terms of the frequencies $\omega$ and $\nu$ the Fourier transforms $\hat{f}$ and $\check{f}$, respectively, and the Fourier integral of the function $f$ (formula for the inverse Fourier transform in $L_2$) have the form

$$f(t) = \int \hat{f}(\omega)e^{i\omega t}d\omega, \quad \text{where} \quad \hat{f}(\omega) = \frac{1}{2\pi} \int f(t)e^{-i\omega t}dt;$$

$$f(t) = \int \check{f}(\nu)e^{2\pi\nu t}d\nu, \quad \text{where} \quad \check{f}(\nu) = \int f(t)e^{-i2\pi\nu t}dt.$$

Since the Fourier transform is invertible, the function $E$ can be recovered from the function $P$ (or $p$). Hence the spectral characteristic or the spectral function $P$ (or $p$) of the device as well as its instrumental function $E$ completely determines the device $A$.

We then calculate $Af$ knowing $P$. Representing $f$ by a Fourier integral we find the representation $Af$ in the form of a Fourier integral:

$$f(t) = \int \hat{f}(\omega)e^{i\omega t}d\omega \quad \text{and} \quad Af(t) = \int \check{f}P(\omega)e^{i\omega t}d\omega.$$

In particular, if $f = \delta$, then