Chapter 8
Entrance Laws and Excursion Laws

The main purpose of this chapter is to investigate the structures of entrance laws for Dawson–Watanabe superprocesses. In particular, we establish a one-to-one correspondence between minimal probability entrance laws for a superprocess and entrance laws for its spatial motion. Based on this result, a complete characterization is given for infinitely divisible probability entrance laws of the superprocess. We also prove some supporting properties of Kuznetsov measures determined by entrance laws. Finally we discuss briefly the special case where the underlying process is an absorbing-barrier Brownian motion in a domain. The results presented here will be used in the study of immigration superprocesses.

8.1 Some Simple Properties

Suppose that $E$ is a Lusin topological space. Let $(Q_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ denote respectively the transition semigroup and the cumulant semigroup of an MB-process with state space $M(E)$. Recall that $(V_t)_{t \geq 0}$ always has the representation (2.5) and $E^\circ$ is the set of points $x \in E$ so that (2.9) holds. Let $(Q_t^\circ)_{t \geq 0}$ denote the restriction of $(Q_t)_{t \geq 0}$ to $M(E)^\circ$.

Theorem 8.1 Given a bounded entrance law $(K_t^\circ)_{t > 0}$ for $(Q_t^\circ)_{t \geq 0}$, we can define a bounded entrance law $(K_t)_{t > 0}$ for $(Q_t)_{t \geq 0}$ by

$$K_t = \lim_{s \to 0} \int_{M(E)^\circ} K_s^\circ(d\mu)Q_{t-s}(\mu, \cdot), \quad t > 0.$$  (8.1)

Moreover, the above relation determines a one-to-one correspondence of bounded entrance laws $(K_t^\circ)_{t > 0}$ for $(Q_t^\circ)_{t \geq 0}$ with bounded entrance laws $(K_t)_{t > 0}$ for $(Q_t)_{t \geq 0}$ satisfying $\lim_{t \to 0} K_t(\{0\}) = 0$.

Proof. If $(K_t^\circ)_{t > 0}$ is a bounded entrance law for $(Q_t^\circ)_{t \geq 0}$, the limit (8.1) clearly exists and defines a bounded entrance law $(K_t)_{t > 0}$ for $(Q_t)_{t \geq 0}$. In fact, $K_t$ is the

entrance laws so that
\[ K_t(\{0\}) = \lim_{s \to 0} K^\circ_s(M(E)^\circ) - K^\circ_t(M(E)^\circ), \]
which implies \( \lim_{t \to 0} K_t(\{0\}) = 0 \). Conversely, if \( (K_t)_{t>0} \) is a bounded entrance law for \( (Q_t)_{t \geq 0} \) satisfying \( \lim_{t \to 0} K_t(\{0\}) = 0 \), we let \( K^\circ_t \) be the restriction of \( K_t \) to \( M(E)^\circ \). It is easy to see that \( (K^\circ_t)_{t>0} \) is a bounded entrance law for \( (Q^\circ_t)_{t \geq 0} \) and (8.1) holds. Then we have the desired one-to-one correspondence.

Theorem 8.2 Let \( K = (K_t)_{t > 0} \) be a family of infinitely divisible probability measures on \( M(E) \) given by
\[ \int_{M(E)} e^{-\nu(f)} K_t(d\nu) = \exp \left\{ - \eta_t(f) - \int_{M(E)^\circ} (1 - e^{-\nu(f)}) H_t(d\nu) \right\}, \quad (8.2) \]
where \( \eta_t \in M(E) \) and \( [1 \land \nu(1)] H_t(d\nu) \) is a finite measure on \( M(E)^\circ \). Then \( K \) is an entrance law for \( (Q_t)_{t \geq 0} \) if and only if
\[ \eta_{r+t} = \int_E \eta_r(dy) \lambda_t(y, \cdot) \quad \text{and} \quad H_{r+t} = \int_E \eta_r(dy) L_t(y, \cdot) + H'_r Q^\circ_{t}, \quad (8.3) \]
for all \( r, t > 0 \).

Proof. By Theorem 1.35 the family of infinitely divisible probability measures \( (K_t)_{t > 0} \) on \( M(E) \) can be represented by (8.2). By Proposition 2.6 one can see (8.3) gives an alternative expression for the relation \( K_{r+t} = K_r Q_t \).

Corollary 8.3 If \( H = (H_t)_{t > 0} \) is a \( \sigma \)-finite entrance law for the restricted semigroup \( (Q^\circ_t)_{t \geq 0} \) satisfying
\[ \int_{M(E)^\circ} [1 \land \nu(1)] H_t(d\nu) < \infty, \quad t > 0, \quad (8.4) \]
then
\[ \int_{M(E)} e^{-\nu(f)} K_t(d\nu) = \exp \left\{ - \int_{M(E)^\circ} (1 - e^{-\nu(f)}) H_t(d\nu) \right\} \quad (8.5) \]
defines an infinitely divisible probability entrance law \( K = (K_t)_{t > 0} \) for \( (Q_t)_{t \geq 0} \).

Corollary 8.4 If \( E^\circ = E \), then (8.5) establishes a one-to-one correspondence between infinitely divisible probability entrance laws \( K \) for \( (Q_t)_{t \geq 0} \) and \( \sigma \)-finite entrance laws \( H \) for \( (Q^\circ_t)_{t \geq 0} \) satisfying (8.4).

We next turn to the special case of a \( (\xi, \phi) \)-superprocess \( X \). Here we assume \( \xi \) is a Borel right process in \( E \) with transition semigroup \( (P_t)_{t \geq 0} \) and \( \phi \) is a branching