1. The Sherrington-Kirkpatrick Model

1.1 Introduction

Consider a large population of individuals (or atoms) that we label from 1 to $N$. Let us assume that each individual knows all the others. The feelings of the individual $i$ towards the individual $j$ are measured by a number $g_{ij}$ that can be positive, or, unfortunately, negative. Let us assume symmetry, $g_{ij} = g_{ji}$, so only the numbers $(g_{ij})_{i<j}$ are relevant. We are trying to model a situation where these feelings are random. We are not trying to make realistic assumptions, but rather to find the simplest possible model; so let us assume that the numbers $(g_{ij})_{i<j}$ are independent random variables. (Throughout the book, the word “independent” should always be understood in the probabilistic sense.) Since we are aiming for simplicity, let us also assume that these random variables (r.v.s) are standard Gaussian. This is the place to point out that Gaussian r.v.s will often be denoted by lower case letters.

A very important feature of this model (called frustration, in physics) is that even if $g_{ij} > 0$ and $g_{jk} > 0$ (that is, $i$ and $j$ are friends, and $j$ and $k$ are friends), then $i$ and $k$ are just as likely to be enemies as they are to be friends. The interactions $(g_{ij})$ describe a very complex social situation.

Let us now think that we fix a typical realization of the numbers $(g_{ij})$. Here and elsewhere we say that an event is “typical” if (for large $N$) it occurs with probability close to 1. For example, the situation where nearly half of the r.v.s $g_{ij}$ are $> 0$ is typical, but the situation where all of them are $< 0$ is certainly not typical. Let us choose the goal of separating the population in two classes, putting, as much as possible, friends together and enemies apart. It should be obvious that at best this can be done very imperfectly: some friends will be separated and some enemies will cohabit. To introduce a quantitative way to measure how well we have succeeded, it is convenient to assign to each individual $i$ a number $\sigma_i \in \{-1, 1\}$, thereby defining two classes of individuals. Possibly the simplest measure of how well these two classes unite friends and separate enemies is the quantity

$$\sum_{i<j} g_{ij} \sigma_i \sigma_j . \tag{1.1}$$
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Trying to make this large invites making the quantities $g_{ij}\sigma_i\sigma_j$ positive, and thus invites in turn taking $\sigma_i$ and $\sigma_j$ of the same sign when $g_{ij} > 0$, and of opposite signs when $g_{ij} < 0$.

Despite the simplicity of the expression (1.1), the optimization problem of finding the maximum of this quantity (for a typical realization of the $g_{ij}$) over the configuration $\sigma = (\sigma_1, \ldots, \sigma_N)$ appears to be of extreme difficulty, and little is rigorously known about it. Equivalently, one can look for a minimum of the function

$$- \sum_{i<j} g_{ij}\sigma_i\sigma_j.$$

Finding the minimum value of a function of the configurations is called in physics a zero-temperature problem, because at zero temperature a system is always found in its configuration of lowest energy. To a zero-temperature problem is often associated a version of the problem “with a temperature”, here the problem corresponding to the Hamiltonian

$$H_N(\sigma) = -\frac{1}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j. \quad (1.2)$$

That is, we think of the quantity (1.2) as being the energy of the configuration $\sigma$. The purpose of the normalization factor $N^{-1/2}$ will be apparent after (1.9) below. The energy level of a given configuration depends on the $(g_{ij})$, and this randomness models the “disorder” of the situation.

The minus signs in the Boltzmann factor $\exp(-\beta H_N(\sigma))$ that arise from the physical requirement to favor configurations of low energy are a nuisance for mathematics. This nuisance is greatly decreased if we think that the object of interest is $(-H_N)$, i.e. that the minus sign is a part of the Hamiltonian. We will use this strategy throughout the book. Keeping with this convention, we write formula (1.2) as

$$- H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i<j} g_{ij}\sigma_i\sigma_j. \quad (1.3)$$

One goal is to understand the system governed by the Hamiltonian (1.3) at a given (typical) realization of the disorder (i.e. the r.v.s $g_{ij}$), or, equivalently, at a given realization of the (random) Hamiltonian $H_N$. To understand better this Hamiltonian, we observe that the energies $H_N(\sigma)$ are centered Gaussian r.v.s. The energies of two different configurations are however not independent. In fact, for two configurations $\sigma^1$ and $\sigma^2$, we have

$$E(H_N(\sigma^1)H_N(\sigma^2)) = \frac{1}{N} \sum_{i<j} \sigma^1_i\sigma^1_j\sigma^2_i\sigma^2_j$$

$$= \frac{N}{2} \left( \frac{1}{N} \sum_{i\leq N} \sigma^1_i\sigma^2_i \right)^2 - \frac{1}{2}. \quad (1.4)$$