Approximation Algorithms for Reliable Stochastic Combinatorial Optimization

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Abstract. We consider optimization problems that can be formulated as minimizing the cost of a feasible solution \( w^T x \) over an arbitrary combinatorial feasible set \( F \subseteq \{0,1\}^n \). For these problems we describe a broad class of corresponding stochastic problems where the cost vector \( W \) has independent random components, unknown at the time of solution. A natural and important objective that incorporates risk in this stochastic setting is to look for a feasible solution whose stochastic cost has a small tail or a small convex combination of mean and standard deviation. Our models can be equivalently reformulated as nonconvex programs for which no efficient algorithms are known. In this paper, we make progress on these hard problems.

Our results are several efficient general-purpose approximation schemes. They use as a black-box (exact or approximate) the solution to the underlying deterministic problem and thus immediately apply to arbitrary combinatorial problems. For example, from an available \( \delta \)-approximation algorithm to the linear problem, we construct a \( \delta (1 + \epsilon) \)-approximation algorithm for the stochastic problem, which invokes the linear algorithm only a logarithmic number of times in the problem input (and polynomial in \( 1/\epsilon \)), for any desired accuracy level \( \epsilon > 0 \). The algorithms are based on a geometric analysis of the curvature and approximability of the nonlinear level sets of the objective functions.

Keywords: Approximation algorithms, reliable optimization, stochastic optimization, risk, mean-risk, nonlinear programming, nonconvex optimization.

1 Introduction

In this paper, we consider generic combinatorial problems and ask what happens when their associated costs are stochastic. The most common approaches in stochastic optimization are to find the solution of minimum expected cost. However, in many applications reliability considerations are very important: risk-averse users need reassurance regarding the level of risk, and not just the expected cost of the provided solution. For example, the transportation community

* This work was supported in part by the National Science Foundation under grant 0931550.
has recognized the importance of reliable route plans (e.g., [27, 24, 36, 9]), however the solutions offered are typically inefficient or heuristic with unknown approximation guarantee. Similarly, reliability is a key consideration in finance and other continuous optimization settings [33]. It has been noted that incorporating reliability [33, 28] transforms the problems into nonconvex ones for which there are no known efficient algorithms and rigorous approximative analysis is scarce. In this paper, we provide a rigorous treatment of reliable combinatorial optimization, offering fully-polynomial approximation schemes for a rich framework of reliability measures.

To illustrate our framework, consider an application such as driving to the airport in uncertain traffic. Our goal is to find a route that gets us to the airport on time. Clearly, the route which minimizes our expected travel time may not be an appropriate choice. In fact, the natural objectives may vary depending on when we are submitting the route query: ahead of time, when we are debating how much time to budget for our trip, or at the start of our trip, when we are optimizing our chance of ontime arrival. In the former setting, we would typically want to allocate enough time to ensure some confidence of ontime arrival, say 95%. In the latter, given a deadline to reach our destination, we need to find the route which will most likely reach by the deadline. Another natural objective, used for example by the Federal Highway Administration as a travel time reliability criterion, is given by the mean plus standard deviation of a route [10]. The latter reliability criterion has been considered in the context of stochastic minimum spanning trees as well [2], and this model is sometimes referred to as mean-risk optimization (e.g., [2]).

We thus focus on a general framework for reliable stochastic combinatorial optimization, which includes the following problem settings:

1. minimize \((\text{mean} + c \cdot \text{standard deviation})\) for a non-negative constant \(c\) which parametrizes the level of risk-aversion. [Call this the \textit{Mean-risk model} or objective.]
2. maximize \(\Pr(\text{solution cost} \leq \text{budget})\) for a given \textit{budget}. [\textit{Probability tail model / objective.}]
3. minimize \textit{budget} such that \(\Pr(\text{solution cost} \leq \text{budget}) \geq p\) for a given confidence probability \(p\). [\textit{Value-at-risk model.}]

In contrast with the diversity in model specifications above, we will show that the same approximation algorithm design can simultaneously address all. Throughout, we assume that the cost distributions are independent, although our algorithms also extend to the case of correlations of neighboring edges for example in shortest path problems (the graph with correlated edges is transformed into a slightly larger graph with independent edges and thus all our results here immediately carry through.)

\textbf{Contributions.} We start our discussion with the (relatively) simpler mean-risk model, which is equivalent to minimizing \((\text{mean} + c \cdot \sqrt{\text{variance}})\). We provide strong results that apply to arbitrary cost distributions with given means and