Chapter 13
The Simplest Supersymmetric Models


The BRST invariance of the classical effective action for the non-Abelian gauge fields is an example of symmetry with parameters of the transformation belonging to the Grassmann algebra. Such symmetries, called supersymmetries, have become increasingly popular in field theory in their own right. Below we present two examples of supersymmetric models: a free field model and the so called Wess–Zumino model.

13.1 Simple Superalgebra

Supersymmetry algebra includes, beside the bosonic generators of the Poincaré group \( P \), at least one spinor generator \( \hat{Q} \). We will discuss in this chapter only the four-dimensional Minkowski space-time and then the simplest possibility is to take \( \hat{Q} \) to be the right-handed Weyl spinor \( \hat{Q}_\alpha, \alpha = 1, 2, \) of the Grassmann type (see Chap. 5). This will lead us to the so called \( N = 1 \) supersymmetry. Our goal will be thus to determine the allowed form of the super-algebra containing—beside the generators of \( P \)—the generator \( \hat{Q}_\alpha \) and its conjugate \( \hat{\bar{Q}}^{\dot{\alpha}} \) (notice that—to conform with most of literature on supersymmetry — we have changed the notation for the conjugate spinor from a ‘star’ to a ‘bar’).

Consider first the commutator \([\hat{P}^{\mu}, \hat{Q}_\alpha]\). It is a spinor quantity, so let us assume that\(^1\):
[\hat{P}^\mu, \hat{Q}_\alpha] = c\sigma^\mu_{\alpha\beta} \hat{Q}^{\dot{\beta}} \quad (13.1)

with some complex constant \(c\). Consequently, upon conjugation of both sides

\[[\hat{P}^\mu, \hat{Q}^{\dot{\beta}}] = -c^* \tilde{\sigma}^\mu \tilde{\beta}^\gamma \hat{Q}_\gamma. \quad (13.2)\]

Using (13.1) and (13.2), the Jacobi identity

\[[\hat{P}^\mu, [\hat{P}^\nu, \hat{Q}_\alpha]] + [\hat{P}^\nu, [\hat{Q}_\alpha, \hat{P}^\mu]] + [\hat{Q}_\alpha, [\hat{P}^\mu, \hat{P}^\nu]] = 0 \quad (13.3)\]

and the relation \([\hat{P}^\mu, \hat{P}^\nu] = 0\), we get

\[|c|^2 (\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu) = 0,\]

so that \(c = 0\), and we have

\[[\hat{P}^\mu, \hat{Q}_\alpha] = [\hat{P}^\mu, \hat{Q}^{\dot{\beta}}] = 0. \quad (13.4)\]

In the spinor representation (see Eq. (5.19)) the Dirac matrices read

\[\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad [\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^\mu\nu & 0 \\ 0 & \tilde{\sigma}^\mu\nu \end{pmatrix}.\]

It then follows from Eq. (5.17) that under the Lorentz transformation with antisymmetric, infinitesimal \(\omega_{\mu\nu}\) the generator \(\hat{Q}_\alpha\) transforms as

\[\hat{Q}'_\alpha = (1 + \frac{1}{2} \omega_{\mu\nu} \sigma^{\mu\nu})_\alpha^\beta \hat{Q}_\beta = \hat{Q}_\alpha + \frac{i}{2} \omega_{\mu\nu} \left[\hat{M}^{\mu\nu}, \hat{Q}_\alpha\right],\]

so that

\[\left[\hat{M}^{\mu\nu}, \hat{Q}_\alpha\right] = -i (\sigma^{\mu\nu})_\alpha^\beta \hat{Q}_\beta. \quad (13.5)\]

Similar derivation gives

\[\left[\hat{M}^{\mu\nu}, \hat{Q}^{\dot{\alpha}}\right] = -i (\tilde{\sigma}^{\mu\nu})^\dot{\alpha}^\dot{\beta} \hat{Q}^{\dot{\beta}}. \quad (13.6)\]

Consider now the anticommutator \([\hat{Q}_\alpha, \hat{Q}^{\dot{\beta}}]\). It is clearly a bosonic object and the transformation properties of \(\hat{Q}_\alpha\) under the Poincaré transformations constrain it to be proportional to \((\sigma^{\mu\nu})_\alpha^\beta \hat{M}^{\mu\nu}\). In view of (13.4)

\[\left[\hat{P}^\mu, [\hat{Q}_\alpha, \hat{Q}^{\dot{\beta}}]\right] = 0.\]