Finitary \( \mathcal{M} \)-Adhesive Categories

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Abstract. Finitary \( \mathcal{M} \)-adhesive categories are \( \mathcal{M} \)-adhesive categories with finite objects only, where the notion \( \mathcal{M} \)-adhesive category is short for weak adhesive high-level replacement (HLR) category. We call an object finite if it has a finite number of \( \mathcal{M} \)-subobjects. In this paper, we show that in finitary \( \mathcal{M} \)-adhesive categories we do not only have all the well-known properties of \( \mathcal{M} \)-adhesive categories, but also all the additional HLR-requirements which are needed to prove the classical results for \( \mathcal{M} \)-adhesive systems. These results are the Local Church-Rosser, Parallelism, Concurrency, Embedding, Extension, and Local Confluence Theorems, where the latter is based on critical pairs. More precisely, we are able to show that finitary \( \mathcal{M} \)-adhesive categories have a unique \( \mathcal{E} \)-\( \mathcal{M} \) factorization and initial pushouts, and the existence of an \( \mathcal{M} \)-initial object implies in addition finite coproducts and a unique \( \mathcal{E}' \)-\( \mathcal{M}' \) pair factorization. Moreover, we can show that the finitary restriction of each \( \mathcal{M} \)-adhesive category is a finitary \( \mathcal{M} \)-adhesive category and finitariness is preserved under functor and comma category constructions based on \( \mathcal{M} \)-adhesive categories. This means that all the classical results are also valid for corresponding finitary \( \mathcal{M} \)-adhesive systems like several kinds of finitary graph and Petri net transformation systems. Finally, we discuss how some of the results can be extended to non-\( \mathcal{M} \)-adhesive categories.

1 Introduction

The concepts of adhesive \([1]\) and (weak) adhesive high-level-replacement (HLR) \([2]\) categories have been a break-through for the double pushout approach (DPO) of algebraic graph transformations \([3]\). Almost all main results in the DPO-approach have been formulated and proven in these categorical frameworks and instantiated to a large variety of HLR systems, including different kinds of graph and Petri net transformation systems. These main results include the Local Church-Rosser, Parallelism, and Concurrency Theorems, the Embedding and Extension Theorem, completeness of critical pairs, and the Local Confluence Theorem.

However, in addition to the well-known properties of adhesive and (weak) adhesive HLR categories \((\mathcal{C}, \mathcal{M})\), also the following additional HLR-requirements have been needed in \([2]\) to prove these main results: finite coproducts compatible with \( \mathcal{M} \), \( \mathcal{E}' \)-\( \mathcal{M}' \) pair factorization usually based on suitable \( \mathcal{E} \)-\( \mathcal{M} \) factorization.
of morphisms, and initial pushouts. It is an open question up to now under which conditions these additional HLR-requirements are valid in order to avoid an explicit verification for each instantiation of an adhesive or (weak) adhesive HLR category. In [4], this has been investigated for comma and functor category constructions of weak adhesive HLR categories, but the results hold only under strong preconditions. In this paper, we close this gap showing that these additional properties are valid in finitary $\mathcal{M}$-adhesive categories. We use the notion “$\mathcal{M}$-adhesive category” as short hand for “weak adhesive HLR category” in the sense of [2]. Moreover, an object $A$ in an $\mathcal{M}$-adhesive category is called finite, if $A$ has (up to isomorphism) only a finite number of $\mathcal{M}$-subobjects, i.e., only finite many $\mathcal{M}$-morphisms $m: A' \to A$ up to isomorphism. The category $\mathcal{C}$ is called finitary, if it has only finite objects. Note, that the notion “finitary” depends on the class $\mathcal{M}$ of monomorphisms and “$\mathcal{C}$ being finitary” must not be confused with “$\mathcal{C}$ being finite” in the sense of a finite number of objects and morphisms. In the standard cases of Sets and Graphs where $\mathcal{M}$ is the class of all monomorphisms, finite objects are exactly finite sets and finite graphs, respectively.

Although in most application areas for the theory of graph transformations only finite graphs are considered, the theory has been developed for general graphs, including also infinite graphs, and it is implicitly assumed that the results can be restricted to finite graphs and to attributed graphs with finite graph part, while the data algebra may be infinite. Obviously, not only Sets and Graphs are adhesive categories but also the full subcategories $\text{Sets}_{\text{fin}}$ of finite sets and $\text{Graphs}_{\text{fin}}$ of finite graphs. But to our knowledge it is an open question whether for each adhesive category $\mathcal{C}$ also the restriction $\mathcal{C}_{\text{fin}}$ to finite objects is again an adhesive category. As far as we know this is true, if the inclusion functor $I: \mathcal{C}_{\text{fin}} \to \mathcal{C}$ preserves monomorphisms, but we are not aware of any adhesive category, where this property fails, or whether this can be shown in general. In this paper, we consider $\mathcal{M}$-adhesive categories $(\mathcal{C},\mathcal{M})$ with restriction to finite objects $(\mathcal{C}_{\text{fin}},\mathcal{M}_{\text{fin}})$, where $\mathcal{M}_{\text{fin}}$ is the restriction of $\mathcal{M}$ to morphisms between finite objects. In this case, the inclusion functor $I: \mathcal{C}_{\text{fin}} \to \mathcal{C}$ preserves $\mathcal{M}$-morphisms, such that finite objects in $\mathcal{C}_{\text{fin}}$ w.r.t. $\mathcal{M}_{\text{fin}}$ are exactly the finite objects in $\mathcal{C}$ w.r.t. $\mathcal{M}$. More generally, we are able to show that the finitary restriction $(\mathcal{C}_{\text{fin}},\mathcal{M}_{\text{fin}})$ of any $\mathcal{M}$-adhesive category $(\mathcal{C},\mathcal{M})$ is a finitary $\mathcal{M}$-adhesive category. Moreover, finitariness is preserved under functor and comma category constructions based on $\mathcal{M}$-adhesive categories.

In Section 2 we introduce basic notions of finitary $\mathcal{M}$-adhesive categories including finite coproducts compatible with $\mathcal{M}$, $\mathcal{M}$-initial objects, finite objects, and finite intersections, which are essential for the theory of finitary $\mathcal{M}$-adhesive categories. The first main result, showing that the additional HLR-requirements mentioned above are valid for finitary $\mathcal{M}$-adhesive categories, is presented in Section 3. In Section 4 we show as second main result that the finitary restriction of an $\mathcal{M}$-adhesive category is a finitary $\mathcal{M}$-adhesive category such that the results of Section 3 are applicable. In Section 5 we show that functorial constructions, including functor and comma categories, applied to finitary $\mathcal{M}$-adhesive