Coxeter Groups and Asynchronous Cellular Automata

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Abstract. The dynamics group of an asynchronous cellular automaton (ACA) relates properties of its long term dynamics to the structure of Coxeter groups. The key mathematical feature connecting these diverse fields is involutions. Group-theoretic results in the latter domain may lead to insight about the dynamics in the former, and vice-versa. In this article, we highlight some central themes and common structures, and discuss novel approaches to some open and open-ended problems. We introduce the state automaton of an ACA, and show how the root automaton of a Coxeter group is essentially part of the state automaton of a related ACA.

Keywords: Asynchronous cellular automaton, Coxeter group, dynamics group, sequential dynamical system.

1 Introduction

An asynchronous cellular automaton (ACA) is defined in the same manner as a classical cellular automaton (CA) in all aspects except the evaluation mechanism. As the name suggests, the maps associated to the vertices (or nodes) are applied synchronously for a CA, and asynchronously for an ACA. In general, there are many ways that one can apply maps asynchronously. For example, one may select a vertex at random according to some probability distribution, apply the corresponding map, and repeat this procedure. Alternatively, one may select a fixed permutation over the vertices and apply the maps in the sequence specified by this permutation. This permutation evaluation process would correspond to increasing the time by one unit, and would be applied repeatedly to generate the system dynamics. An important aspect of having a fixed permutation update sequence is that one obtains a dynamical system. This is not necessarily the case in the more general situation, such as when the individual states are updated at random.

The analysis of CAs and ACAs does not have the support that the study of ODEs has from established fields such as analysis and differential geometry. As such, a key goal of CA/ACA research is to make connections to existing mathematical theory. We will consider the class of $\pi$-independent ACAs – those whose periodic points (as a set) are independent of the permutation update sequence. While this may seem to be a rather exotic property, we have shown that
roughly 40% of the elementary CA rules give rise to $\pi$-independent ACAs \[8\].

Given a $\pi$-independent ACA, one can define its \textit{dynamics group}. This permutation group on the set of periodic points is a quotient of a Coxeter group, and it captures the possible long-term dynamics that one can generate by suitable choices of update sequence. Its structure can answer questions about the existence and non-existence of periodic orbits of given sizes.

In this paper, we will revisit the notions of Coxeter systems and sequential dynamical systems (SDSs). An SDS is a generalization of an ACA (assuming a fixed update sequence) where the underlying graph is arbitrary, and is not limited to being a regular lattice or circle (i.e., a one-dimensional torus). We will show how the word problem for Coxeter groups is related to functional equivalence of SDS maps. This forms the basis for our next result, on how conjugation of Coxeter elements corresponds to cycle equivalence of SDS maps, and additionally, how this extends from conjugacy classes to spectral classes. After defining dynamics groups and showing how they arise as quotients of Coxeter groups, we show how key features of mathematical objects in both the fields of SDSs and Coxeter groups are encoded by finite (or infinite) state automata. We illustrate this by explicit examples, and then close with a table summarizing these connections.

2 Background

A \textit{Coxeter system} is a pair $(W, S)$ consisting of a group $W$ generated by a set $S = \{s_1, \ldots, s_n\}$ of involutions given by the following presentation

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_is_j)^{m(s_i,s_j)} = 1 \rangle,$$

where $m(s_i,s_j) \geq 2$ for $i \neq j$. Let $S^*$ be the free monoid over $S$, and for each integer $m \geq 0$ and distinct generators $s, t \in S$, define

$$\langle s, t \rangle_m = \underbrace{stst \cdots}_{m} \in S^*.$$

The relation $\langle s, t \rangle_{m(s,t)} = \langle t, s \rangle_{m(s,t)}$ is called a \textit{braid relation}, and is additionally called a \textit{short braid relation} if $m(s,t) = 2$. Note that $s$ and $t$ commute if and only if $m(s,t) = 2$. A Coxeter system can be described uniquely by its \textit{Coxeter graph} $\Gamma$, which has vertex set $V = \{1, \ldots, n\}$ and an edge $\{i, j\}$ for each non-commuting pair of generators $\{s_i, s_j\}$, with edge label $m(s_i,s_j)$.

Switching to ACAs and SDSs, let $\Gamma$ be an undirected graph (called the \textit{base graph} or \textit{dependency graph}) with vertex set $V = \{1, \ldots, n\}$. We equip each vertex $i$ with a state $x_i \in K$ where $K$ is a set called the \textit{state space}, and a \textit{vertex function} $f_i$ that maps (or updates) $x_i(t)$ to $x_i(t+1)$ based on the states of its neighbors (itself included). Unless explicitly stated otherwise, we will assume that $K = \mathbb{F}_2 = \{0, 1\}$, which is the most commonly used state space in cellular automata research. If the vertex functions are applied asynchronously, it is convenient to encode $f_i$ as a $\Gamma$-local function $F_i : K^n \to K^n$ defined by

$$F_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, f_i(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n).$$