7 Binomial Model for European Options

A large range of options exist for which the boundary conditions of the Black-Scholes differential equation are too complex to solve analytically; an example being the American option. One therefore has to rely on numerical price computation. The best known methods for this is to approximate the stock price process by a discrete time stochastic process, or, as in the approach followed by Cox, Ross, Rubinstein, model the stock price process as a discrete time process from the start. By doing this, the options time to maturity $T$ is decomposed into $n$ equidistant time steps of length

$$\Delta t = \frac{T}{n}.$$ 

We consider therefore the discrete time points

$$t_j = j \Delta t, \ j = 0, \ldots, n.$$ 

By $S_j$ we denote the stock price at time $t_j$. At the same time, we discretize the set of values the stock price can take, such that it takes on many finite values $S_j^k, k = 1, \ldots, m_j$, with $j$ denoting the point of time and $k$ representing the value. If the stock price is in time $t_j$ equal to $S_j^k$, then it can jump in the next time step to one of $m_j+1$ new states $S_{j+1}^l, \ l = 1, \ldots, m_j+1$. The probabilities associated to these movements are denoted by $p_{kl}^j$:

$$p_{kl}^j = \text{P}(S_{j+1}^l = S_{j+1}^l | S_j = S_j^k),$$

with

$$\sum_{l=1}^{m_j+1} p_{kl}^j = 1, \ 0 \leq p_{kl}^j \leq 1.$$ 

If we know the stock price at the current time, we can build up a tree of possible prices up to a certain point in time, for example the maturity date $T = t_n$. Such a tree is also called stock price tree. Should the option price be known at the final point in time $t_n$ of the stock price tree, for example by means of the options intrinsic value, the option value at time $t_{n-1}$ can
be computed (according to (6.24)) as the discounted conditional expectation of the corresponding option prices at time $t_n$ given the stock price at time $t_{n-1}$:

$$V(S_{n-1}^k, t_{n-1}) = e^{-r \Delta t} \mathbb{E}\{V(S_n, t_n) \mid S_{n-1} = S_{n-1}^k\}$$

$$= e^{-r \Delta t} \sum_{l=1}^{m_n} p_{kl}^{n-1} V(S_l, t_n).$$  \hspace{1cm} (7.1)

$V(S, t)$ again denotes the option value at time $t$ if the underlying has a price of $S$. Repeating this step for the remaining time steps $t_j$, $j = n-2, n-3, \ldots, 0$, means that the option prices up to time $t = 0$ can be approximated.

### 7.1 Cox–Ross–Rubinstein Approach to Option Pricing

As the simplest example to price an option, we consider the approach by Cox, Ross and Rubinstein (CRR) which is based on the assumption of a binomial model, and which can be interpreted as a numerical method to solve the Black–Scholes equation. We will look at European options exclusively and assume, for the time being, that the underlying pays no dividends within the time to maturity. Again, we discretize time and solely consider the points in time $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots, t_n = n\Delta t = T$ with $\Delta t = \frac{T}{n}$. The binomial model proceeds from the assumption that the discrete time stock price process $S_j$ follows a geometric random walk (see Chapter 4), which is the discrete analogue of the geometric Brownian motion. The binomial model has the special feature that at any point in time the stock price has only two possibilities to move:

- either the price moves at rate $u$ and with probability $p$ in one direction (for example it moves up)
- or the price moves at rate $d$ and with probability $1 - p$ in another direction (for example it moves down).

Using the notation introduced above, if the stock price in time $t_j$ is equal to $S_j^k$ then in time $t_{j+1}$ it can take only the values $u \cdot S_j^k$ and $d \cdot S_j^k$. The probabilities $p$ and $q$ are independent of $j$. All other probabilities $p_{kl}$ associated to $S_{j+1}^l \neq u \cdot S_j^k$ and $\neq d \cdot S_j^k$ are 0.

In order to approximate the Black–Scholes differential equation by means of the Cox–Ross–Rubinstein approach, the probabilities $p, q$ as well as the rates $u, d$ have to be chosen such that in the limit $\Delta t \to 0$ the binomial model