Approximating Probability Measures on Manifolds via Radial Basis Functions

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Summary. Approximating a given probability measure by a sequence of normalized counting measures is an interesting problem and has broad applications in many areas of mathematics and engineering. If the target measure is the uniform distribution on a manifold then such approximation gives rise to the theory of uniform distribution of point sets and the corresponding discrepancy estimates. If the target measure is the equilibrium measure on a manifold, then such approximation leads to the minimization of certain energy functionals, which have applications in discretization of manifolds, best possible site selection for polynomial interpolation and Monte Carlo method, among others. Traditionally, polynomials are the major tool in this arena, as have been demonstrated in the celebrated Weyl's criterion, Erdős-Turán inequalities. Recently, the novel approach of employing radial basis functions (RBFs) has been successful, especially in higher dimensional manifolds. In its general methodology, RBFs provide an efficient vehicle that allows a certain type of linear translation operators to act in various function spaces, including reproducing kernel Hilbert spaces (RKHS) associated with RBFs. This approach is crucial in the establishment of the LeVeque type inequalities that are capable of giving discrepancy estimates for some minimal energy configurations. We provide an overview of the recent developments outlined above. In the final section we show that many results on the sphere can be generalised to other compact homogeneous manifolds. We also propose a few research topics for future investigation in this area.

1 Introduction

Let M be a d-dimensional manifold embedded in $\mathbb{R}^m$ $(m \geq d)$. Let a probability measure ν be given on M. Let $N \in \mathbb{N} \setminus \{1\}$. We are interested in finding a set of N distinct points $x_1, \ldots, x_N$ in M such that the normalized counting measure

$$\sigma_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \tag{1}$$

approximates ν well according to a given criterion which will be specified later. Here δ_{x_j} denotes the unit mass at the point x_j. Note that we have eliminated

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the interesting (but trivial) case $N = 1$, in which the unit mass δ_{x_1} at the *center of gravity* x_1 (may or may not be in M) of the measure ν is often the undisputable choice.

Lets first consider the simple example in which we are approximating the uniform distribution μ on the interval $[0, 1)$. Note that the density function of μ is the constant function: $t \mapsto 1, \;\; t \in [0, 1)$. Let a triangular array

$$\{x_{N,1}, \ldots, x_{N,N}\}_{N=2}^{\infty}$$

be given. We say that the set $\{x_{N,1}, \ldots, x_{N,N}\}$ is uniformly distributed in $[0, 1)$ (as $n \to \infty$) if for each fixed $0 < x < 1$, we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{x_{N,j} : j = 1, \ldots, N, x_{N,j} \in [0, x)\} = x.$$

We remark that the above limit is equivalent to the following:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \chi_{[0,x)}(x_{N,j}) = \int_0^1 \chi_{[0,x)}(t)dt = x,$$

where $\chi_{[0,x)}$ is the indicator function of the interval $[0, x)$. The collection of the indicator functions $\{\chi_{[0,x)} : x \in [0, 1)\}$ plays an important role here. They provide a *testing ground* for the approximation of the uniform distribution by a sequence of normalized counting measures. In his study of uniform distribution of points, Weyl [47] used trigonometrical polynomials to approximate these indicator functions, and obtained the celebrated Weyl's criterion that asserts that the set $\{x_{N,1}, \ldots, x_{N,N}\}$ is uniformly distributed in $[0, 1)$ (as $n \to \infty$) if and only if for each integer k, $k \neq 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i k x_{N,j}} = 0.$$

Weyl's criterion can be considered as a qualitative characterization of uniform distribution of point sets. To measure uniform distribution of point sets in a quantitative way, one needs the notion of "discrepancy". There are many different (but similar) ways of defining discrepancy. For the time being, we use the so called "star" discrepancy $D^*(N)(\sigma_1, \sigma_2)$ between the two probability measures σ_1, σ_2 on the interval $[0, 1)$ defined by

$$D^*(\sigma_1, \sigma_2) := \sup_{x\in[0,1)} \left| \int_0^1 \chi_{[0,x)}(t)(d\sigma_1(t) - d\sigma_2(t)) \right|.$$

The star discrepancy $D^*(N)(\sigma_N, \mu)$ will be simply denoted by $D^*(N)$.

Erdős and Turán [11] refined Weyl's trigonometrical polynomial approximation scheme and proved the following theorem that has since been called the Erdős-Turán Inequality: