Beauty depends on size as well as symmetry.

Aristotle (384–322 BCE), Poetics

Characteristic of Weyl was an aesthetic sense which dominated his thinking on all subjects. He once said to me, half-joking, “My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful.” (Hermann Weyl (1885–1955))

F. Dyson, in Nature, March 10, 1956

This section is devoted to several specific examples of theorems and configurations in projective geometry. Clearly, our considerations in Chapter 5 demonstrated that there is an infinite variety of incidence theorems in real projective geometry. Any polynomial identity like $(x + y)^2 = x^2 + 2 \cdot x \cdot y + y^2$ can be translated into an incidence theorem via von Staudt constructions. For this, one models every elementary addition or multiplication by a suitable subconfiguration. The equality in the equation translates to a final coincidence of two lines that forms the conclusion of the theorem. Figure 15.1 shows a suitable geometric construction for the equation above (lines that appear to be parallel are assumed to be parallel).

Clearly, most of the incidences obtained in this way will be very boring, since they represent only more or less trivial facts about algebraic expressions. In fact, the simplest algebraic expressions will surprisingly lead to nicer geometric theorems than the complicated ones. For instance, we saw in Section 5.7 that Pappos’s theorem expresses the equation $x \cdot y = y \cdot x$. 
Fig. 15.1 An incidence theorem from \((x + y)^2 = x^2 + 2 \cdot x \cdot y + y^2\).

It is unreasonable to try to find a formal notion of *mathematical beauty*, but nevertheless many geometric theorems are considered to be beautiful mathematical objects. Typically there are several (interrelated) ingredients that make a geometric theorem a nice result:

- *simplicity* (it is easy to state the theorem),
- *symmetry* (are there repeated patterns),
- *surprising conclusions* (is the conclusion somehow unexpected),
- *size* (the fewer hypotheses the better),
- *generalizability* (does the theorem extend to a whole class of objects).

In this section we want to explore several theorems and configurations that are expressible in terms of projective geometry. In particular, we will explore relations to bracket expressions and explore their combinatorial symmetries and the symmetries of the corresponding bracket expressions. We will also use this section to demonstrate several techniques that are useful in relating geometric scenarios to algebraic expressions. Several times we will provide different proofs for the same facts in order to demonstrate different approaches. I hope the reader will share the appreciation of the mathematical beauty of many of these structures.\(^1\)

### 15.1 Desargues’s Theorem

The first theorem we will meet is an incidence theorem similar to Pappos’s theorem. However, in contrast, it requires 10 points and 10 lines. Each point

\[^1\text{Some of the proofs presented here already occurred in the introductory Chapter 1, where we explored different approaches to Pappos's theorem. For matters of completeness they are also included in this chapter.}\]