4

Lines and Cross-Ratios

At this stage of this monograph we enter a significant didactic problem. There are three concepts that are intimately related and that unfold their full power only if they play together. These concepts are performing calculations with geometric objects, determinants and determinant algebra, and geometric incidence theorems. The reader should understand that in a beginner’s text that makes few assumptions about prior knowledge, these concepts must be introduced sequentially. Therefore we will sacrifice some mathematical beauty for clarity of exposition. Still, we highly recommend that the following chapters be read (at least) twice, so that the reader may obtain an impression of the interplay of the different concepts.

This and the next section are dedicated to the relationship between $\mathbb{R}P^2$ and calculations in the underlying field $\mathbb{R}$. For this we will first find methods to relate points in a projective plane to the coordinates over $\mathbb{R}$. Then we will show that elementary operations like addition and multiplication can be mimicked in a purely geometric fashion. Finally, we will use these facts to derive interesting statements about the structure of projective planes and projective transformations.
4.1 Coordinates on a Line

Assume that two distinct points \([p]\) and \([q]\) in \(\mathcal{P}_\mathbb{R}\) are given. How can we describe the set of all points on the line through these two points? It is clear that we can implicitly describe them by first calculating the homogeneous coordinates of the line through \(p\) and \(q\) and then selecting all points that are incident to this line. However, there is also a very direct and explicit way of describing these points, as the following lemma shows:

**Lemma 4.1.** Let \([p]\) and \([q]\) be two distinct points in \(\mathcal{P}_\mathbb{R}\). The set of all points on the line through these points is given by

\[
\{[\lambda \cdot p + \mu \cdot q] \mid \lambda, \mu \in \mathbb{R} \text{ with } \lambda \text{ or } \mu \text{ nonzero} \}.
\]

**Proof.** The proof is an exercise in elementary linear algebra. For \(\lambda, \mu \in \mathbb{R}\) (with \(\lambda\) or \(\mu\) nonzero) let \(r = \lambda \cdot p + \mu \cdot q\) be a representative of a point. We have to show that this point is on the line through \([p]\) and \([q]\). In other words, we must prove that \(\langle \lambda \cdot p + \mu \cdot q, p \times q \rangle = 0\). This is an immediate consequence of the arithmetic rules for the scalar and vector products. We have

\[
\langle \lambda \cdot p + \mu \cdot q, p \times q \rangle = \langle \lambda \cdot p, p \times q \rangle + \langle \mu \cdot q, p \times q \rangle = \lambda \cdot 0 + \mu \cdot 0 = 0.
\]

The first two equations hold by multilinearity of the scalar product. The third equation comes from the fact that \(\langle p, p \times q \rangle\) and \(\langle q, p \times q \rangle\) are always zero.

Conversely, assume that \([r]\) is a point on the line spanned by \([p]\) and \([q]\). This means that there is a vector \(l \in \mathbb{R}^3\) with

\[
\langle l, p \rangle = \langle l, q \rangle = \langle l, r \rangle = 0.
\]

The points \([p]\) and \([q]\) are distinct, and thus \(p\) and \(q\) are linearly independent. Consider the matrix \(M\) with row vectors \(p, q, r\). This matrix cannot have full rank, since the product \(M \cdot l\) is the zero vector. Thus \(r\) must lie in the span of \(p\) and \(q\). Since \(r\) itself is not the zero vector, we have a representation of the form \(r = \lambda \cdot p + \mu \cdot q\) with \(\lambda\) or \(\mu\) nonzero.

The last proof is simply an algebraic version of the geometric fact that we consider a line as the linear span of two distinct points on it. In the form \(r = \lambda \cdot p + \mu \cdot q\) we can simultaneously multiply both parameters \(\lambda\) and \(\mu\) by the same factor \(\alpha\) and still obtain the same point \([r]\). If one of the two parameters is nonzero we can normalize this parameter to 1. Using this fact we can express almost all points on the line through \([p]\) and \([q]\) by the expression \(\lambda \cdot p + q\); \(\lambda \in \mathbb{R}\). The only point we miss is \([p]\) itself. Similarly,