Chapter 9

Extending research on patterns in permutations and words to other domains

In this section we discuss several extensions of research on patterns in permutations and words to other contexts. Among these extensions are a requirement on parity of elements (see Section 9.1), considerations of multisets (see Section 9.3) and signed/colored permutations (see Section 9.6), increasing the number of dimensions (see Subsections 9.8.2 and 9.8.3), and some other directions/generalizations.

Note that unlike the previous chapters, this chapter is neither thorough in the sense of definitions/examples, nor comprehensive in listing the results in question. However, additional information can be obtained by following the references we provide.

9.1 Refining patterns/statistics

The statistic “descent” is refined in [526] by Kitaev and Remmel, by fixing the parity of (exactly) one of the descent’s two letters. Let $E = \{0, 2, 4, \ldots\}$ and $O = \{1, 3, 5, \ldots\}$ denote the set of even and odd numbers, respectively. Given $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, we define the following notions, where $\chi(\sigma_1 \in X)$ is 1 if $\sigma_1$ is of type $X$, and it is 0 otherwise.

- $\overleftarrow{\text{Des}}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} & \sigma_i \in X\}; \overleftarrow{\text{des}}_X(\sigma) = |\overleftarrow{\text{Des}}_X(\sigma)|$ for $X \in \{E, O\}$;
- $\overrightarrow{\text{Des}}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} & \sigma_{i+1} \in X\}; \overrightarrow{\text{des}}_X(\sigma) = |\overrightarrow{\text{Des}}_X(\sigma)|$ for $X \in \{E, O\}$;

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\[ \sum_{\sigma \in S_n} x_{\text{deg}}(\sigma) = \sum_{k=0}^{n} R_{k,n} x^k; \]
\[ \sum_{\sigma \in S_n} x_{\text{des}}(\sigma) = \sum_{k=0}^{n} M_{k,n} x^k; \]
\[ \sum_{\sigma \in S_n} x_{\text{des}}(\sigma) \gamma(\sigma_1 \in E) = \sum_{k=0}^{n} \sum_{j=0}^{1} P_{j,k,n} z^j x^k; \]
\[ \sum_{\sigma \in S_n} x_{\text{des}}(\sigma) \gamma(\sigma_1 \in O) = \sum_{k=0}^{n} \sum_{j=0}^{1} Q_{j,k,n} z^j x^k. \]

The following explicit formulas for the distribution of the four new statistics are obtained in [526] using differential operators.

**Theorem 9.1.1. ([526])** We have

- \( R_{k,2n} = \binom{n}{k}^2 (n!)^2; \)
- \( R_{k,2n+1} = (k+1) \binom{n}{k+1}^2 (n!)^2 + (2n+1-k) \binom{n}{k}^2 (n!)^2 = \frac{1}{k+1} \binom{n}{k}^2 ((n+1)!)^2; \)
- \( P_{1,k,2n} = \binom{n-1}{k} \binom{n}{k+1} (n!)^2; \)
- \( P_{1,k,2n+1} = (n+1) \frac{(n-k)}{k+1} \binom{n}{k}^2 (n!)^2; \)
- \( P_{0,k,2n} = \binom{n-1}{k} \binom{n}{k} (n!)^2; \)
- \( P_{0,k,2n+1} = (k+1) \binom{n}{k+1} (n!)^2 = (n+1) \binom{n}{k}^2 (n!)^2; \)
- \( Q_{0,k,2n} = \binom{n-1}{k-1} \binom{n}{k} (n!)^2; \)
- \( Q_{0,k,2n+1} = \frac{(n+1)(n-k+1)}{k} \binom{n}{k}^2 (n!)^2; \)
- \( Q_{1,k,2n} = \binom{n-1}{k} \binom{n}{k} (n!)^2; \)
- \( Q_{1,k,2n+1} = \binom{n}{k}^2 n!(n+1)!; \)
- \( M_{k,2n} = \frac{n+1}{k+1} \binom{n-1}{k} \binom{n}{k} (n!)^2 = \binom{n-1}{k} \binom{n+1}{k+1} (n!)^2; \)
- \( M_{k,2n+1} = \frac{1}{n-k+1} \binom{n}{k}^2 ((n+1)!)^2 = \binom{n}{k} \binom{n+1}{k} n!(n+1)!; \)

Distribution of descents according to parity can be viewed as distribution of consecutive occurrences of the following patterns. We use \( e, o, \) or \( * \) as superscripts for a pattern’s letters to require that in an occurrence of the pattern in a permutation, the corresponding letters must be even, odd or either. For example, the permutation 25314 has two occurrences of the pattern 2\^e1^o (they are 53 and 31, both of them are