Period Matrices of Polyhedral Surfaces

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7.1 Introduction

Finding a conformal parameterization for a surface and computing its period matrix is a classical problem which is useful in a lot of contexts, from statistical mechanics to computer graphics.

The 2D-Ising model [Mer01,CSM02,CSM03] for example can be realized on a cellular decomposition of a surface whose edges are decorated by interaction constants, understood as a discrete conformal structure. In certain configurations, called critical temperature, the model exhibits conformal invariance properties in the thermodynamical limit and certain statistical expectations become discrete holomorphic at the finite level. The computation of the period matrix of higher genus surfaces built from the rectangular and triangular lattices from discrete Riemann theory has been addressed in the cited papers by Costa-Santos and McCoy.

Global conformal parameterization of a surface is important in computer graphics [JWYG04,DMA02,BCGB08,TACSD06,KSS06,War06,SSP08] in issues such as texture mapping of a flat picture onto a curved surface in \( \mathbb{R}^3 \). When the texture is recognized by the user as a natural texture known as featuring round grains, these features should be preserved when mapped onto the surface, mainly because any shear of circles into ellipses is going to be wrongly interpreted as suggesting depth increase. Characterizing a surface by a few numbers is as well a desired feature in computer graphics, for problems like pattern recognition. Computing numerically the period matrix of a surface has been addressed in [JWYG04].

This chapter uses the general framework of the theory of discrete Riemann surfaces [Fer44,Duf68,Mer01,BMS05] and the computation of period matrices within this framework (based on theorems and not only numerical analogies). Other approaches have emerged recently [GN07,DN03,Kis08]. We focus here on the computation of period matrices within this framework (based on theorems and not only numerical analogies).
We start with some surfaces with known period matrices and compute numerically their discrete period matrices, at different levels of refinement. In particular, some genus two surfaces made up of squares and the Wente torus are considered. We observe numerically good convergence properties. Moreover, we compute the yet unknown period matrix of the Lawson surface, identify it numerically as one of the tested surfaces, which allows us to conjecture their conformal equivalence, and finally to prove it.

7.2 Discrete Conformal Structure

Consider a polyhedral surface in $\mathbb{R}^3$. It has a unique Delaunay tessellation, generically a triangulation [BS07]. That is to say each face is associated with a circumcircle drawn on the surface and this disk contains no other vertices than the ones on its boundary. Let’s call $\Gamma$ the graph of this cellular decomposition, $\Gamma_0$ its vertices, $\Gamma_1$ its edges and complete it into a cellular decomposition with $\Gamma_2$ the set of triangles. Each edge $(x, x') = e \in \Gamma_1$ is adjacent to a pair of triangles, associated with two circumcenters $y, y'$. The ratio of the (intrinsic) distances between the circumcenters and the length of the (orthogonal) edge $e$ is called $\rho(e)$. It is the celebrated cotan formula [PP93].

Following [Mer01], we call these data of a graph $\Gamma$, whose edges are equipped with a positive real number, a discrete conformal structure. A discrete Riemann surface is a conformal equivalence class of surfaces with the same discrete conformal structure. It leads to a theory of discrete Riemann surfaces and discrete analytic functions, developed in [Fer44, Duf68, Mer01, Mer04, Mer07, BMS05, DKT08], that shares a lot of features with the continuous theory, and these features are recovered in a proper refinement limit. We are going to summarize these results (Fig. 7.1).

In our examples, the extrinsic triangulations are Delaunay. That is to say the triangulations come from the embedding in $\mathbb{R}^3$ and the edges $(x, x') \in \Gamma_1$ of the triangulation are the edges of the polyhedral surface in $\mathbb{R}^3$. On the other hand, the geodesic connecting the circumcenters $y$ and $y'$ on the surface is not an interval of a straight line and its length is generically greater than the distance $||y - y'||$ in $\mathbb{R}^3$. The latter gives a more naive definition of $\rho$ and is