3 Functions, Sequences and Limits

In this chapter we define the notations of function and sequence and introduce the most important concept of calculus: the limit. This chapter is vital to the understanding of any further reading; in particular, the reader must come to control the subject of limit and convergence.

3.1 Introduction

A preliminary notation, necessary for introducing the function concept, is the Cartesian product.

Definition 3.1.1. The Cartesian product of given sets \( A \) and \( B \) is the set \( A \times B \) of all the ordered pairs \( (a,b) \), where \( a \in A \) and \( b \in B \):

\[
A \times B = \{(a,b) \mid a \in A, b \in B\}
\]

Example 3.1.1. Let \( A = \{1,2,4\} \), \( B = \{2,3\} \). Then

\[
A \times B = \{(1,2),(1,3),(2,2),(2,3),(4,2),(4,3)\}
\]

Definition 3.1.2. Any subset of the \( A \times B \) is called a relation from \( A \) to \( B \). The set of all the ordered pairs of a relation present the graph or the curve of the relation.

Example 3.1.2. In the previous example, the set \( \{(1,2),(2,2),(2,3),(4,3)\} \) is a relation from \( A \) to \( B \). In order to obtain the graph associated with this relation, we first draw two infinitely long straight lines, perpendicular to each other. These horizontal and vertical lines are called axes and are usually denoted by \( x \) and \( y \) respectively. Secondly, we mark all the points corresponding to the ordered pairs, relative to the intersection point of the axes (Fig. 3.1.1), called origin and defined as \( (0,0) \).
Example 3.1.3. The following sets are relations from $\mathbb{R}$ to $\mathbb{R}$:

(a) $S_1 = \{(x, y) \mid x \in \mathbb{R}, y = x^3\}$
(b) $S_2 = \{(x, y) \mid x, y \in \mathbb{R}, x^2 + y^2 = 1\}$

In case (a) we calculate $x^3$ for every $x \in \mathbb{R}$ and place the ordered pair $(x, x^3)$ in $S_1$. In case (b), given an arbitrary $x \in \mathbb{R}$, we find all the $y$’s such that $x^2 + y^2 = 1$ and place each $(x, y)$ in $S_2$.

Throughout this text we will usually discuss relations where $A$ and $B$ are subsets of $\mathbb{R}$. The most commonly used subsets are the intervals, which are defined next.

Definition 3.1.3. The following subsets of $\mathbb{R}$ are called intervals:

\[
\begin{align*}
[a, b] &\equiv \{ x \mid a \leq x \leq b, a, b \in \mathbb{R} \} \quad \text{(closed interval with endpoints $a$ and $b$)} \\
(a, b) &\equiv \{ x \mid a < x < b, a, b \in \mathbb{R} \} \quad \text{(open interval with endpoints $a$ and $b$)} \\
[a, b) &\equiv \{ x \mid a \leq x < b, a, b \in \mathbb{R} \} \quad \text{(semi open interval with endpoints $a$ and $b$)} \\
(a, b] &\equiv \{ x \mid a < x \leq b, a, b \in \mathbb{R} \}
\end{align*}
\]

A semi open interval is also called ‘semi closed’ or ‘half open (closed)’. The symbol ‘\equiv’ means “defined as”.

For example, the set of all the real numbers between 2 and 17.5 (including 2 and 17.5) is the closed interval $[2,17.5]$. We often rewrite the set of all the real numbers, $\mathbb{R}$, as $\mathbb{R} = \{ x \mid -\infty < x < \infty \}$ or as $\mathbb{R} = (-\infty, \infty)$. By $(-\infty, a)$ we mean “all the real numbers less than $a$”. By $(b, \infty)$ we mean “all the real numbers greater than $b$”. If the endpoint is included we write $(-\infty, a]$ and $[b, \infty)$. Note that the symbols $-\infty$ (minus infinity) and $\infty$ (infinity) are not legitimate real numbers.