1 Models for heavy-tailed asset returns

Szymon Borak, Adam Misiorek, and Rafał Weron

1.1 Introduction

Many of the concepts in theoretical and empirical finance developed over the past decades – including the classical portfolio theory, the Black-Scholes-Merton option pricing model or the RiskMetrics variance-covariance approach to Value at Risk (VaR) – rest upon the assumption that asset returns follow a normal distribution. But this assumption is not justified by empirical data! Rather, the empirical observations exhibit excess kurtosis, more colloquially known as fat tails or heavy tails (Guillaume et al., 1997; Rachev and Mittnik, 2000). The contrast with the Gaussian law can be striking, as in Figure 1.1 where we illustrate this phenomenon using a ten-year history of the Dow Jones Industrial Average (DJIA) index.

In the context of VaR calculations, the problem of the underestimation of risk by the Gaussian distribution has been dealt with by the regulators in an ad hoc way. The Basle Committee on Banking Supervision (1995) suggested that for the purpose of determining minimum capital reserves financial institutions use a 10-day VaR at the 99% confidence level multiplied by a safety factor \( s \in [3, 4] \). Stahl (1997) and Danielsson, Hartmann and De Vries (1998) argue convincingly that the range of \( s \) is a result of the heavy-tailed nature of asset returns. Namely, if we assume that the distribution is symmetric and has finite variance \( \sigma^2 \) then from Chebyshev’s inequality we have \( \Pr(\text{Loss} \geq \epsilon) \leq \frac{1}{2\epsilon^2} \sigma^2 \). Setting the right hand side to 1% yields an upper bound for VaR\(_{99}\% \leq 7.07\sigma\). On the other hand, if we assume that returns are normally distributed we arrive at VaR\(_{99}\% \leq 2.33\sigma\), which is roughly three times lower than the bound obtained for a heavy-tailed, finite variance distribution.
1 Models for heavy-tailed asset returns

Being aware of the underestimation of risk by the Gaussian law we should consider using heavy-tailed alternatives. This chapter is intended as a guide to such models. In Section 1.2 we describe the historically oldest heavy-tailed model – the stable laws. Next, in Section 1.3 we briefly characterize their recent lighter-tailed generalizations, the so-called truncated and tempered stable distributions. In Section 1.4 we study the class of generalized hyperbolic laws, which – like tempered stable distributions – can be classified somewhere between infinite variance stable laws and the Gaussian distribution. Finally, in Section 1.5 we provide numerical examples.

1.2 Stable distributions

1.2.1 Definitions and basic properties

The theoretical rationale for modeling asset returns by the Gaussian distribution comes from the Central Limit Theorem (CLT), which states that the sum of a large number of independent, identically distributed (i.i.d.) variables – say, decisions of investors – from a finite-variance distribution will be (asympt-