Probabilistic Büchi Automata with Non-extremal Acceptance Thresholds

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Abstract. This paper investigates the power of Probabilistic Büchi Automata (PBA) when the threshold probability of acceptance is non-extremal, i.e., is a value strictly between 0 and 1. Many practical randomized algorithms are designed to work under non-extremal threshold probabilities and thus it is important to study power of PBAs for such cases.

The paper presents a number of surprising expressiveness and decidability results for PBAs when the threshold probability is non-extremal. Some of these results sharply contrast with the results for extremal threshold probabilities. The paper also presents results for Hierarchical PBAs and for an interesting subclass of them called simple PBAs.

1 Introduction

Probabilistic Büchi Automata (PBA), introduced in [2] to model open, reactive probabilistic systems, are finite state machines that process input strings of infinite length like Büchi automata. However, unlike Büchi automata, they have probabilistic transitions. The semantics of such machines is defined as follows. A run on an input word is considered to be accepting if it satisfies the Büchi acceptance condition. The collection of all accepting runs on any input is known to be measurable [14,2]. For any given acceptance threshold \( x \), the language \( L_{\geq x}(B) \) (\( L_{> x}(B) \)) of a PBA \( B \) is defined to be the set of all inputs for which the above measure is \( \geq x \) (\( > x \)).

In a series of papers [2,19,4], researchers have studied the behavior of PBAs when the acceptance threshold \( x \) is either 0 or 1, delineating the expressive power of such machines and establishing the precise complexity of various decision problems. While extremal thresholds (of 0 and 1) are important for studying randomized algorithms and protocols, in many practical scenarios only algorithms with non-extremal thresholds can solve the problem — consensus in synchronized distributed systems [15], and semantic security [8], being a couple of examples. Thus, studying PBAs under non-extremal thresholds, which is the focus of this paper, is important.

We begin by observing that for non-extremal thresholds \( x \in (0, 1) \), the actual value of \( x \) itself is not important: for every PBA \( B \), one can efficiently construct another PBA \( B' \) such that \( L_{> x}(B) = L_{> \frac{1}{2}}(B') \) (or \( L_{\geq x}(B) = L_{\geq \frac{1}{2}}(B') \)). Thus, we consider the acceptance threshold to be always \( \frac{1}{2} \). Our results on the decidability of the emptiness and universality decision problems are summarized in Figure 1.

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A few salient points about our results on decision problems are as follows. Typically, solving decision problems for automata with non-extremal thresholds is harder than for those with extremal thresholds, as is borne out by similar results for probabilistic finite automata and for finite state probabilistic monitors. Interestingly, this observation does not hold for checking emptiness of $L_{>0}(B)$ for a given PBA $B$, but holds for other problems. More specifically, for a given PBA $B$, the problems of checking emptiness of $L_{>\frac{1}{2}}(B)$ and emptiness of $L_{>0}(B)$ have the same level of undecidability; both of them are $\Sigma^0_2$-complete. On the other hand, the problems of checking emptiness and universality of $L_{\geq \frac{1}{2}}(B)$ are $\Pi^1_1$-complete and co-R.E.-complete, respectively, as opposed to both being $\text{PSPACE}$-complete for $L_{=1}(B)$. The universality problem for $L_{>\frac{1}{2}}(B)$ is $\Pi^1_1$-complete as opposed to being $\Sigma^0_2$-complete for $L_{>0}(B)$.

Previously, we [4] had introduced a syntactic subclass of PBAs called hierarchical PBAs (HPBA) as an expressively less powerful, but computationally more tractable fragment of PBAs. With extremal thresholds, the emptiness and universality problems are efficiently decidable — emptiness and universality of $L_{>0}(B)$ are $\text{NL}$-complete and $\text{PSPACE}$-complete, respectively, while for $L_{=1}(B)$ they are $\text{PSPACE}$-complete and $\text{NL}$-complete, when $B$ is an HPBA. Considering non-extremal acceptance thresholds, these decision problems not only become undecidable, but are as difficult as in the case general PBAs. The only exception to this is the case of checking emptiness of $L_{>\frac{1}{2}}(B)$ which is co-R.E.-complete when $B$ is an HPBA and is $\Sigma^0_2$-complete for general PBAs. This upper bound of co-R.E. in this case is established by observing that for an HPBA $B$, $L_{>\frac{1}{2}}(B)$ is non-empty if and only if there is an ultimately periodic word in $L_{>\frac{1}{2}}(B)$; this observation may be of independent interest.

Next, our undecidability proofs for these various decision problems rely on Condon and Lipton’s ideas, used to show the undecidability of the emptiness problem of probabilistic finite automata. However, in order to obtain lower bounds for HPBAs and obtain “hierarchical” machines, we modify the original reduction by Condon and Lipton, and we believe our modification yields a conceptually simpler proof of the undecidability of the emptiness problem for probabilistic finite automata. In order, to prove the undecidability result, Condon and Lipton do the following. Given a 2-counter machine $M$, they construct a probabilistic finite automata $A_M$ whose inputs are computations of $M$, such that a correct halting computation of $M$, repeated sufficiently many times, is accepted by $A_M$ with high probability ($>\frac{1}{2}$) and all other inputs are rejected with high probability. Thus, $L_{>\frac{1}{2}}(A_M)$ is non-empty iff $M$ has a halting computation. Now, in order to carry out this reduction, the automaton $A_M$ “checks” every pair of successive configurations in the input for correctness, and maintains a variety of bounded counters to ensure that the asymptotic probability of acceptance has the desired properties. We observe that if the automaton only “checks” one pair of successive configurations (where the pair to be checked is chosen randomly) the reduction still works, yielding a “simpler” automaton construction and a simpler analysis of the asymptotics. However, one casualty of our simpler proof is the following — while we can show that the emptiness problem of probabilistic finite automata is undecidable, the Condon-Lipton proof establishes a stronger fact, namely, that the problem remains undecidable even under the promise that the acceptance probability of every input is bounded away from $\frac{1}{2}$.