Abstract. Let $2S_4 \ast Q_8$ be the central product of a double cover $2S_4$ of the symmetric group $S_4$ and the quaternion group $Q_8$. We consider the Galois embedding problem given by $2S_4 \ast Q_8$ as a double cover of the direct product $S_4 \times V_4$ of the symmetric group $S_4$ and the Klein group $V_4$ over a field $K$ of characteristic different from 2. If 2 or $-2$ is a square in $K$, we give a general formula for the solutions to this embedding problem, whenever it is solvable, in terms of quadratic forms. This result answers a question raised by Abhyankar.

1 Introduction

During the workshop on “Algorithmic Number Theory” held in Dagstuhl in July 99, Professor Abhyankar raised the question of obtaining a general method of resolution for the Galois embedding problem given by a double cover $2S_4 \ast Q_8$ of the group $S_4 \times V_4$ over a field of characteristic different from 2. The aim of this paper is to give an answer to Abhyankar’s question.

Let us first recall the definitions and fix notation. We denote by $2S_n$ one of the two double covers of the symmetric group $S_n$ reducing to the non trivial double cover $2A_n$ of the alternating group $A_n$ and by $Q_8$ the quaternion group, which is a double cover of the Klein group $V_4$. The group $2S_4 \ast Q_8$ is the central product of $2S_4$ and $Q_8$. In the sequel $K$ will denote a field of characteristic different from 2. If $L|K$ is a Galois extension with Galois group the direct product $S_4 \times V_4$, we consider the embedding problem

$$2S_4 \ast Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K).$$

A solution to this embedding problem is an extension field $\tilde{L}$ of the field $L$ which is a Galois extension of $K$ with Galois group $2S_4 \ast Q_8$ and such that the restriction epimorphism between the Galois groups agrees with the epimorphism $2S_4 \ast Q_8 \rightarrow S_4 \times V_4$. If $\tilde{L} = L(\sqrt{r})$ is a solution, then the general solution is $L(\sqrt{r^2 r})$, $r \in K^\ast$.  

* Partially supported by BFM2000-0794-C02-01, Spanish Ministry of Education.
2 Preliminaries

Let be given two disjoint Galois extensions of the field $K$, $L_1$ with Galois group $S_4$ and $L_2$ with Galois group $V_4$, let $L = L_1 \cdot L_2$ be the composition. Assume that $L_1$ is the splitting field of a degree 4 polynomial $P(X) \in K[X]$ and $L_2 = K(\sqrt{a}, \sqrt{b})$, $a, b \in K$. Let us consider the double covers $2S_4 \rightarrow S_4$ and $Q_8 \rightarrow V_4$ and let $\varepsilon_1 \in H^2(S_4, \{\pm 1\})$, $\varepsilon_2 \in H^2(V_4, \{\pm 1\})$ denote the corresponding cohomology elements. Let $\pi_1 : S_4 \times V_4 \rightarrow S_4$ and $\pi_2 : S_4 \times V_4 \rightarrow V_4$ be the two projections and $\pi_1^*, \pi_2^*$ the induced morphisms between the 2-cohomology groups. Then the element $\varepsilon = \pi_1(\varepsilon_1).\pi_2(\varepsilon_2) \in H^2(S_4 \times V_4, \{\pm 1\})$ corresponds to the double cover $2S_4 \ast Q_8$ of $S_4 \times V_4$. This implies that the obstruction to the solvability of the embedding problem

$$2S_4 \ast Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)$$  \hspace{1cm} (1)

is equal to the product of the obstructions to the solvability of the embedding problems $2S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$ and $Q_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$. Let us now specify notation by writing $2^+S_n$ or $2^-S_n$ depending on whether transpositions in $S_n$ lift in the double cover to involutions or to elements of order 4. By [3] the obstruction to the solvability of the embedding problem $2^+S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$ is equal to $w(Q_E).(2, d) \in H^2(G_K, \{\pm 1\})$, where $E$ denotes the subextension of $L_1$ obtained by adjoining to $K$ one root of the polynomial $P$, $Q_E$ and $d = d_E$ are the trace form and the discriminant of this extension, $w$ denotes the Hasse-Witt invariant of a quadratic form and $(..)$ a Hilbert symbol. With the same notations, the obstruction to the solvability of the embedding problem $2^-S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$ is equal to $w(Q_E).(-2, d) \in H^2(G_K, \{\pm 1\})$. By [4], the obstruction to the solvability of $Q_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$ is equal to $(a, b).(-1, ab) \in H^2(G_K, \{\pm 1\})$. Therefore the embedding problem $2^\pm S_4 \ast Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)$ is solvable if and only if

$$w(Q_E) = (\pm 2, d).(a, b).(-1, ab) \in H^2(G_K, \{\pm 1\}).$$ \hspace{1cm} (2)

We note that if both embedding problems $2S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$ and $Q_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$ are solvable and $L_1(\sqrt{\gamma_1})$, $L_2(\sqrt{\gamma_2})$ are solutions to them, then $L(\sqrt{\gamma_1 \gamma_2})$ is a solution to (1). Our method of resolution of (1) in the general case is based on the observation that the group $2^\pm S_4 \ast Q_8$ is the pullback of the diagram $S_4 \times V_4 \hookrightarrow S_8 \leftarrow 2^\pm S_8$, where the embedding of $S_4 \times V_4$ in the symmetric group $S_8$ is obtained by making $S_4$ acting on the first four letters and $V_4$ transitively on the last four ones. We can then solve the considered embedding problem by the method given in [2].

This method is based on the fact that the solvability of the considered embedding problem implies the existence of a $K$-graded isomorphism between Clifford algebras. In order to obtain an explicit expression for such an isomorphism and to reach to an element $\gamma$ providing the solutions to the considered embedding problem, we will use the following lemmas.