Meshfree methods are recent and modern discretization techniques for numerically solving partial differential equations (PDEs). In contrast to the well-established traditional methods, such as finite differences (FD), finite volumes (FV), and finite element methods (FEM), meshfree methods do not require sophisticated algorithms and data structures for maintaining a grid, which is often the most time consuming task in mesh-based simulations.

Moreover, meshfree methods provide flexible, robust and reliable discretizations, which are particularly suited for multiscale simulation. Consequently, meshfree discretizations have recently gained much attention in many different applications from computational sciences and engineering, as well as in numerical analysis. Among a few others, the currently most prominent meshfree discretization techniques are smoothed particle hydrodynamics (SPH) [123], the partition of unity method (PUM) [5, 120], and radial basis functions (RBF), see Chapter 3.

In this chapter, a novel adaptive meshfree method of (backward) characteristics, AMMoC, for multiscale simulation of transport processes is proposed. The method AMMoC combines an adaptive Lagrangian particle method with local scattered data interpolation by polyharmonic splines. The adaption strategy is built on customized rules for the refinement and coarsening of current nodes, each at a time corresponding to one flow particle. The required adaption rules are constructed by using available results concerning local error estimates and numerical stability of polyharmonic spline interpolation, as discussed in Section 3.8 of Chapter 3.

The outline of this chapter is as follows. In the following Section 6.1, a short discussion on transport equations is provided, before the basic ingredients of our method AMMoC are explained in Section 6.2. The construction of adaption rules is discussed in Section 6.3. Finally, Section 6.4 is devoted to numerical simulation of various multiscale phenomena in flow modelling.

The model problems discussed in Section 6.4 comprise tracer transportation over the artic, Burgers equation, and two-phase fluid flow in porous medium. The latter is done by using the five-spot problem, a popular model problem from hydrocarbon reservoir simulation, where AMMoC is shown to be competitive with two leading commercial reservoir simulators, ECLIPSE and FrontSim of Schlumberger.
6.1 Transport Equations

Many physical phenomena in transport processes are described by time-dependent hyperbolic conservation laws. Their governing equations have the form

$$\frac{\partial u}{\partial t} + \nabla f(u) = 0$$

(6.1)

where for some domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and a compact time interval $I = [0, T]$, $T > 0$, the unknown function $u : I \times \Omega \to \mathbb{R}$ corresponds to a physical quantity, such as saturation or concentration density. Moreover, the function $f(u) = (f_1(u), \ldots, f_d(u))^T$ in (6.1) denotes the flux tensor.

In this chapter, we consider numerically solving (6.1) on given initial conditions

$$u(0, x) = u_0(x), \quad \text{for } x \in \Omega = \mathbb{R}^d,$$

(6.2)

where we assume that the solution $u$ of the resulting initial value problem has compact support in $I \times \mathbb{R}^d$. The latter serves to avoid considering explicit boundary conditions.

In situations where the flux tensor $f$ is a linear function, i.e.,

$$f(u) \equiv \mathbf{v} \cdot \mathbf{u},$$

(6.3)

we obtain the linear advection equation

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = 0$$

(6.4)

provided that the given velocity field

$$\mathbf{v} = \mathbf{v}(t, x), \quad t \in I, \ x \in \Omega,$$

is divergence-free, i.e.,

$$\text{div} \mathbf{v} = \sum_{j=1}^{d} \frac{\partial v_j}{\partial x_j} \equiv 0.$$

This special case of (6.1) is also often referred to as passive advection. In this case, the scalar solution $u$ is constant along the streamlines of (flow) particles, and the shapes of these streamlines are entirely and uniquely determined by the velocity field $\mathbf{v}$.

In previous work [10], an adaptive meshfree method for solving linear advection equations of the above form is proposed. The method in [10] is a combination of an adaptive version of the semi-Lagrangian method (SLM) and the meshfree radial basis function interpolation (RBF). The resulting advection scheme is then in [11] used for the simulation of tracer transport in the arctic stratosphere. Selected details on this challenging model problem are explained later in this chapter, see Subsection 6.4.1.