Chapter 10
Probabilistic fracture mechanics

10.1 Introduction

The failure analysis of a structure proceeds on the basis of a fracture or failure criterion. A typical example is the criterion of brittle fracture $K_I = K_{Ic}$ which states that failure does not take place for $K_I < K_{Ic}$. Application of such a criterion in the deterministic sense requires all involved quantities to be exactly known. This, however, is not always the case. For instance, the in-service loading conditions of a technical component as well as the material’s fracture toughness $K_{Ic}$ may scatter. Also the location, size, and orientation of cracks is sometimes not precisely known. If these details are neglected and only ‘averaged’ quantities are employed the deterministic analysis may lead to rather vague results. If, on the other hand, the fluctuations are accounted for by considering an upper bound for $K_I$ and a lower bound for $K_{Ic}$ one obtains results which might be safe but probably are too conservative. It also has to be noted that these bounds likewise may not be exactly known. In any way, the risk of fracture remains unknown in the framework of a deterministic analysis. The same holds for all other failure criteria such as the classical failure hypotheses discussed in Chapter 2 or the life-time hypothesis according to the Paris law (Section 4.11).

In contrast to the deterministic analysis, the probabilistic approach takes into account the scatter and uncertainties of material properties, loading conditions, and defect distribution in an appropriate manner. Thereby it is assumed that the quantities entering a failure criterion are given in terms of probability distributions. This leads to statements with respect to the failure probability which determines the risk of fracture.

Statistical aspects also come into play when microstructural features of a material, which are relevant for its fracture behavior, are to be accounted for. In real materials usually a multitude of ‘defects’ such as microvoids, microcracks, inclusions, or inhomogeneities of different size, shape, and orientation are found which have a strong influence on the fracture process. Because of their large number the
effect of these defects on the macroscopic behavior is suitably described by means of statistical methods.

The present chapter deals only with the basic concepts of probabilistic fracture mechanics. It is restricted to brittle materials the strength properties of which may display especially strong scatter. Brittle materials often also show a pronounced decrease in strength with increasing volume of a testing specimen. The reason for this is the distribution of defects: the probability for the occurrence of a critical defect increases with the volume under consideration. This is the foundation of the statistical theory of brittle fracture developed by W. Weibull. In many situations it is employed for the assessment of the behavior of ceramics, fiber-reinforced materials, geological materials, concrete, or brittle metals.

### 10.2 Foundations

The frequency by which some quantity \( x \) occurs, for instance, the measured \( K_{Ic} \) value of a material or a crack length, is described by the probability density \( f(x) \) (Fig. 10.1). If we assume that \( x \) attains only positive values the probability distribution is given by

\[
F(x) = \int_0^x f(\bar{x})d\bar{x}.
\]  

(10.1)

It determines the probability \( P \) that a random variable \( X \) lies in the interval \( 0 \leq X \leq x \):

\[
P(X \leq x) = F(x).
\]  

(10.2)

Since \( P \) can attain values between 0 and 1 the following relations hold:

\[
P(X < \infty) = \int_0^\infty f(x)dx = 1,
\]

\[
P(X \geq x) = 1 - F(x),
\]  

(10.3)

\[
P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a).
\]

The mean (or expectation) value \( \langle X \rangle \) of a random variable and the variance \( \text{var}X \) are defined as

\[
\langle X \rangle = \int_0^\infty x f(x)dx = \int_0^\infty [1 - F(x)]dx,
\]  

(10.4)

\[
\text{var}X = \int_0^\infty [x - \langle X \rangle]^2 f(x)dx.
\]