6 Finite Poles and Zeros of LTV Systems

6.1 Introduction

Stability of an LTV system can be evaluated from the stability of its autonomous part. This is shown in Chapter 12 where an analytic approach for stability of the LTV systems is given. However, stability can be studied using the poles of the system. This is the direction followed in the present section.

The approaches which establish for LTV systems a relation between poles and stability can be mainly split into two classes: the ones in the first class define the poles as the roots of the factors of decompositions of the differential operator associated with the system while for the approaches in the second class, the system is transformed, using operations which preserve stability, into another system which has a state matrix of upper triangular form; the poles are then defined as the diagonal terms. An approach which belongs to the first class is presented in this section, while the second class is discussed in Chapter 12.

The concept of poles of a system has received special attention as a mean to evaluate stability. Although this relation is clear and well known for linear time-invariant systems, the time-varying case is much more difficult since the frozen-time (pointwise-in-time) eigenanalysis may not contain information about stability as it is the case in the following example.

Example 964. Consider the second order Euler equation

\[ \ddot{y} + 0.25t^{-1}\dot{y} + 0.01t^{-2}y = 0 \] (6.1)

which can be written \( P(\partial)y = 0 \) with \( P(\partial) = \partial^2 + 0.25t^{-1}\partial + 0.01t^{-2}. \) The roots of polynomial \( P(\partial) \) computed in a classical manner - called frozen roots as they are parameterized by \( t \), the variable time - are \(-0.2t^{-1} \) and \(-0.05t^{-1} \). Both roots have negative real parts for all \( t \geq t_0 > 0. \) However, the system given by (6.1) is unstable. Indeed, a fundamental set of solutions of (6.1) is \( y_1(t) = t^\frac{3}{4} + \frac{\sqrt{209}}{40}, \) \( y_2(t) = t^\frac{3}{4} - \frac{\sqrt{209}}{40} \) and \( \lim_{t \to +\infty} y_1(t) = +\infty. \)
Intrinsic definitions which overcome the difficulties pointed out above are given in this chapter. First, in Section 6.3 we clarify the connection between the roots of the factors of the skew polynomial which defines the autonomous part of the LTV system and the trajectories of the system. Next, in Section 6.6 the poles and the zeros of the system are defined using the module framework developed in Chapter 5. The usual relations between the poles and zeros are investigated in Section 6.6.5. Section 6.4 shows how field extensions which make it possible to factorize skew polynomials can be constructed (Those skew polynomials define the autonomous parts of the systems we are interested in).

6.2 Exponential Stability

Consider an autonomous LTV system $\Sigma$ and let $T$ be the associated torsion module. $T$ can be given by a n-th order differential equation in one variable in the following polynomial form

$$P(\partial)y = 0, P(\partial) = \partial^n + a_{n-1}\partial^{n-1} + \ldots + a_1\partial + a_0, a_i \in K \tag{6.2}$$

where $P$ is a skew polynomial with coefficients in the differential field $K$ (see Theorem and Definition 662(1), Lemma and Definition 781 (2)(v) and Example 964).

Let $R = K[\partial; \delta]$ ($\delta = d/dt$) and consider the $\mathbb{C}$-space $W = \mathcal{O}_\infty$ introduced in (5.4.2.2). Recall that any element $y$ of $\mathcal{O}_\infty$ can be viewed as an analytic function $(a, +\infty) \ni t \mapsto y(t) \in \mathbb{C}$ where $a$ is large enough (Example 20). Assume that $W$ is an injective cogenerator of $R\text{Mod}$. This happens if $K$ is one of the differential fields considered in (5.4.2.2) according to Theorem 838.

A solution of (6.2) in $W = \mathcal{O}_\infty$ is an element $y \in \text{Hom}_R(T, W)$ (Corollary 567; Lemma and Definition 771(i)).

**Definition 965.** The autonomous LTV system $\Sigma$ given by the torsion module $T$ is said

- exponentially stable if any solution $y \in \text{Hom}_R(T, W) : t \mapsto y(t)$ of (6.2) approaches zero exponentially for $t \to +\infty$, i.e., if there exist constants $\alpha > 0$, $\beta > 0$ and $t_0 > 0$ such that

$$\|y(t)\| \leq \alpha e^{-\beta(t-t_0)}, t \geq t_0. \tag{6.3}$$

- exponentially unstable if there exists a solution $y : t \mapsto y(t)$ of (6.2) which is exponentially unbounded, i.e., if there exist constants $\alpha > 0$, $\beta > 0$ and $t_0 > 0$ such that

$$\|y(t)\| \geq \alpha e^{\beta(t-t_0)}, t \geq t_0. \tag{6.4}$$

**Remark 966.** For short, the ring of differential operators $R = K[\partial; \delta]$ is denoted by $K[\partial]$ in the sequel.