Chapter 1
Introduction to Fractional Calculus

1.1 Introduction

Fractional calculus is three centuries old as the conventional calculus, but not very popular amongst science and engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity). Thereby this considers the history and non-local distributed effects. In other words perhaps this subject translates the reality of nature better! Therefore to make this subject available as a popular subject to science and engineering community, adds another dimension to understand or describe basic nature in a better way. Perhaps fractional calculus is what nature understands and to talk with nature in this language is therefore efficient. For past three centuries this subject was with mathematicians and only in last few years, this is pulled to several (applied) fields of engineering and science and economics. However recent attempt is on to have definition of fractional derivative as a local operator specifically to fractal science theory. Next decade will see several applications based on this three hundred years (old) new subject, which can be thought of as supersets of fractional differintegrals, the conventional integer order calculus being a part of it. Differintegration is operator doing differentiation and sometimes integrations in a general sense. Also the applications and discussions are limited to fixed fractional order differintegrals and the variable order of differintegration is kept as future research subject. Perhaps the Fractional Calculus will be the calculus of XXI century. In this book attempt is made to make this topic application oriented for regular science and engineering applications. Therefore rigorous mathematics is kept minimal. In this introductory chapter list in tabular form is provided to readers to have feel of fractional derivatives of some commonly occurring functions. Here various definitions of fractional differ-integrations are introduced, and some applications of Fractional Differential Equations are deliberated. Also, discussed are the law of irreversibility, non-locality and conservation of probability, stable random variables, scale invariance and generalization of normal probability density function to power law distributions. The simple derivation is shown here to elucidate the fact that integer order diffusion equation Fick’s law has half order differentiation embedded into it, and its extension to have concept of anomalous transport, and Fractional Brownian Motion. This treatment will give the readers the feel of fractional
calculus to explain natural laws, stochastic processes of several manifestations and
the subject’s relevance to basic physical laws of nature. To start with let us think for a
moment, about the normal derivative \( \frac{d}{dt} \) is representing the rate of accumulation
or loss; that is, gain rate minus loss rate, at infinitesimal bounded space. Well if that
infinitesimal space is having traps (of various sizes) where our particle money or any
variable what we are studying is temporarily parked then will the \( \frac{d}{dt} \) replicate real
picture of accumulation or loss? Similarly these trap pictures could be islands or for-
bidden zones in the infinitesimal space where the variable (particle, money, flux etc.)
cannot reside; then also the rate of accumulation or loss will be different
than \( \frac{d}{dt} \). Well then the fractional differentiation \( \frac{d^\alpha}{dt^\alpha} \), where \( \alpha \in \mathbb{R} \) may
give the sub- or super-rate of accumulation or loss with index \( \alpha \) representing the
heterogeneity distribution of the infinitesimal space (traps or islands)!

1.2 Birth of Fractional Calculus

In a letter dated 30th September 1695, L’Hopital wrote to Leibniz asking him particu-
lar notation he has used in his publication for the \( n \)-th derivative of a function

\[
\frac{D^n f(x)}{Dx^n}
\]

i.e. what would the result be if \( n = 1/2 \). Leibniz’s response “an apparent paradox
from which one day useful consequences will be drawn.” In these words fractional
calculus was born. Studies over the intervening three hundred years have proven at
least half right. It is clear, that with in the XX century, especially numerous applica-
tions have been found. However, these applications and mathematical background
surrounding fractional calculus are far from paradoxical. While the physical meaning
is difficult to grasp, the definitions are no more rigorous than integer order counterpart.

Later the question became: Can \( n \), be any number: fractional, irrational, or com-
plex? Because the latter question was answered affirmatively, the name ‘fractional
calculus’ has become a misnomer and might better be called ‘integration and differ-
entiation of arbitrary order’ or ‘arbitrary ordered differ-integrations’. Anyway mathe-
ematics is art of giving misleading names!

In 1812, P.S. Laplace defined a fractional derivative of arbitrary order appeared in
Lacroix’s (1819) writings. He developed a mere mathematical exercise generalizing
from a case of integer order. Starting with \( y = x^m \), where \( m \) a positive integer, La-
croix easily develops \( n \) th derivative:

\[
\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n.
\]

Using Legengdre’s symbol for the generalized factorial (the complete Gamma
function), Lacroix gets: