Appendix B

Application of the Markov Model to Life Insurance

B.1 Traditional Rating of Life Contracts

Before starting with the Markov model, I would like to summarise how traditional calculations using commutation functions are performed. Usually one starts with the probabilities of death and then calculates a decrement table starting with, say, 100000 persons at age 20.

After that one, has to calculate the different commutation functions, which I assume everybody knows by heart. These numbers depend on the persons alive and on the technical interest rate \( i \). Only when you have done this it is (in the classical framework) possible to calculate the necessary premiums. In the following we will look a little bit closer at the calculation of a single premium for an annuity. To do this we need the following commutation functions:

\[
D_x = v \times l_x \text{ where } l_x \text{ denotes the number of persons alive at age } x.
\]

\[
C_x = v \times (l_{x+1} - l_x)
\]

Having this formalism it is well known that

\[
\ddot{a}_x = \frac{N_x}{D_x}
\]

From this example is easily seen that almost all premiums can be calculated by summation and multiplication of commutation functions. Such an approach has its advantages in an environment where calculations have to be performed by hand, or where computers are expensive. Calculation becomes messy if benefits are considered with guarantees or with refunds.
The Markov model here presented offers rating of life contracts without using commutation functions. It starts with calculation of the reserves and uses the involved probabilities directly. In order to see such a calculation let's review the above-mentioned example: We will use $n p_x$ to denote the probability of a person aged exactly $x$ surviving for $n$ years.

$$\ddot{a}_x = \sum_{j=0}^{\infty} j p_x \times v^j$$

$$= 1 + p_x \times \ddot{a}_{x+1}$$

The above formula gives us a recursion for the mathematical reserves of the contract. Hence one can calculate the necessary single premiums just by recursion. In order to do this, we need an initial condition, which is in our case $V_{\omega} = 0$.

The interpretation of the formula is easy: The necessary reserve at age $x$ consists of two parts:

1. The annuity payment, and
2. The necessary reserve at age $x+1$. (These reserves must naturally be discounted.)

It should be pointed out that the calculation does not need any of the commutation functions; only $p_x$ and the discount factor $v$ are used. As a consequence this method does not produce the overheads of traditional methods.

In the following paragraphs the discrete time, discrete state Markov model is introduced and solutions of some concrete problems are offered.

At this point, it is necessary to stress the fact that the following frame work can be used, with some modifications, in an environment with stochastic interest. But as we are limited in space and time we have to restrict ourselves to deterministic constant discount rates.

### B.2 Life Insurance Considered as Random Cash Flows

The starting point of the Markov model is a set of states, which correspond to the different possible conditions of the insured persons. In life insurance the set of states usually consists of alive, dead. The set of states will be denoted by $\mathcal{S}$.

The second point which originates from the life contract has to do with the so-called contractual functions which depend on the states and the time. Hence the structure of a generalised life contract can be thought of:

Contractual situation between time $t$ and time $t + 1$