Bounds on Complexity of Tests, Decision Rules and Trees

In this chapter, we continue the consideration of decision tables with one-valued decisions. We study bounds on complexity for decision trees, rules, and tests.

The chapter consists of three sections. In Sect. 3.1 we investigate lower bounds on the depth of decision trees, cardinality of tests and length of decision rules.

Section 3.2 is devoted to the consideration of upper bounds on the minimum cardinality of tests and minimum depth of decision trees. These bounds can be used also as upper bounds on the minimum length of decision rules.

Section 3.3 contains conclusions.

3.1 Lower Bounds

From Corollaries 2.24 and 2.28 it follows that \( L(T) \leq h(T) \leq R(T) \). So each lower bound on \( L(T) \) is a lower bound on \( h(T) \) and \( R(T) \), and each lower bound on \( h(T) \) is also a lower bound on \( R(T) \).

Let us consider now some lower bounds on the value \( h(T) \) and, consequently, on the value \( R(T) \).

We denote by \( D(T) \) the number of different decisions in a decision table \( T \).

**Theorem 3.1.** Let \( T \) be a nonempty decision table. Then

\[
h(T) \geq \log_2 D(T).
\]

**Proof.** Let \( \Gamma \) be a decision tree for \( T \) such that \( h(\Gamma) = h(T) \). We denote by \( L_t(\Gamma) \) the number of terminal nodes in \( \Gamma \). It is clear that \( L_t(\Gamma) \geq D(T) \). One can show that \( L_t(\Gamma) \leq 2^{h(\Gamma)} \). Therefore \( 2^{h(\Gamma)} \geq D(T) \) and \( h(\Gamma) \geq \log_2 D(T) \). Thus, \( h(T) \geq \log_2 D(T) \).

\[\square\]
Theorem 3.2. Let $T$ be a decision table. Then

$$h(T) \geq \log_2 (R(T) + 1).$$

Proof. Let $\Gamma$ be a decision tree for $T$ such that $h(\Gamma) = h(T)$. We denote by $L_w(\Gamma)$ the number of working nodes in $\Gamma$. From Theorem 2.23 it follows that the set of attributes attached to working nodes of $\Gamma$ is a test for $T$. Therefore $L_w(\Gamma) \geq R(T)$. One can show that $L_w(\Gamma) \leq 1 + 2 + \ldots + 2^{h(\Gamma)} - 1 = 2^{h(\Gamma)} - 1$. Therefore $2^{h(\Gamma)} - 1 \geq R(T)$. Since $h(\Gamma) = h(T)$ we obtain $h(T) \geq \log_2 (R(T) + 1)$. $\Box$

Example 3.3. Let us consider the decision table $T$ depicted in Fig. 3.1.

For this table $D(T) = 3$. Using Theorem 3.1 we obtain $h(T) \geq \log_2 3$. Therefore $h(T) \geq 2$.

One can show that this table has exactly two tests: $\{f_1, f_2, f_3\}$ and $\{f_2, f_3\}$. Therefore $R(T) = 2$. Using Theorem 3.2 we obtain $h(T) \geq \log_2 3$ and $h(T) \geq 2$.

In fact, $h(T) = 2$. A decision tree for the table $T$ which depth is equal to 2 is depicted in Fig. 3.2.

Let $T$ be a decision table with $n$ columns which are labeled with attributes $f_1, \ldots, f_n$. A subtable of the table $T$ is a table obtained from $T$ by removal some rows. Let $\{f_{i_1}, \ldots, f_{i_m}\} \in \{f_1, \ldots, f_n\}$ and $\delta_1, \ldots, \delta_m \in \{0, 1\}$. We denote by $T(f_{i_1}, \delta_1) \ldots (f_{i_m}, \delta_m)$ the subtable of the table $T$ which consists of rows that at the intersection with columns $f_{i_1}, \ldots, f_{i_m}$ have numbers $\delta_1, \ldots, \delta_m$. 