Chapter 4

Connections and Curvature

4.1 Connections in Vector Bundles

Let $X$ be a vector field on $\mathbb{R}^d$, $V$ a vector at $x_0 \in \mathbb{R}^d$. We want to analyze how one takes the derivative of $X$ at $x_0$ in the direction $V$. For this derivative, one forms

$$
\lim_{t \to 0} \frac{X(x_0 + tV) - X(x_0)}{t}.
$$

Thus, one first adds the vector $tV$ to the point $x_0$. Next, one compares the vector $X(x_0 + tV)$ at the point $x_0 + tV$ and the vector $X(x_0)$ at $x_0$; more precisely, one subtracts the second vector from the first one. Division by $t$ and taking the limit then are obvious steps.

A vector field on $\mathbb{R}^d$ is a section of the tangent bundle $T(\mathbb{R}^d)$. Thus, $X(x_0 + tV)$ lies in $T_{x_0 + tV}(\mathbb{R}^d)$, while $X(x_0)$ lies in $T_{x_0}(\mathbb{R}^d)$. The two vectors are contained in different spaces, and in order to subtract the second one from the first one, one needs to identify these spaces. In $\mathbb{R}^d$, this is easy. Namely, for each $x \in \mathbb{R}^d$, $T_x\mathbb{R}^d$ can be canonically identified with $T_0\mathbb{R}^d \cong \mathbb{R}^d$. For this, one uses Euclidean coordinates and identifies the tangent vector $\frac{\partial}{\partial x_i}$ at $x$ with $\frac{\partial}{\partial x_i}$ at $0$. This identification is even expressed by the notation. The reason why it is canonical is simply that the Euclidean coordinates of $\mathbb{R}^d$ can be obtained in a geometric manner. For this, let $c(t) = tx$, $t \in [0, 1]$ be the straight line joining $0$ and $x$. For a vector $X_1$ at $x$, let $X_t$ be the vector at $c(t)$ parallel to $X_1$; in particular, $X_t$ has the same length as $X_1$ and forms the same angle with $\dot{c}$. $X_0$ then is the vector at $0$ that gets identified with $X_1$. The advantage of the preceding geometric description lies in the fact that $X_1$ and $X_0$ are connected through a continuous geometric process. Again, this process in $\mathbb{R}^d$ has to be considered as canonical.
On a manifold, in general there is no canonical method anymore for identifying tangent spaces at different points, or, more generally fibers of a vector bundle at different points. For example, on a general manifold, we don’t have canonical coordinates. Thus, we have to expect that a notion of derivative for sections of a vector bundle, for example for vector fields, has to depend on certain choices.

Definition 4.1.1. Let $M$ be a differentiable manifold, $E$ a vector bundle over $M$. A covariant derivative, or equivalently, a (linear) connection is a map

$$D : \Gamma(E) \rightarrow \Gamma(E) \otimes \Gamma(T^*M)$$

with the properties subsequently listed:

By property (i) below, we may also consider $D$ as a map from $\Gamma(TM) \otimes \Gamma(E)$ to $\Gamma(E)$ and write for $\sigma \in \Gamma(E), V \in T_x M$

$$D\sigma(V) =: D_V \sigma.$$ 

We then require:

(i) $D$ is tensorial in $V$ :

$$D_{V+W}\sigma = D_V\sigma + D_W\sigma \quad \text{for } V, W \in T_x M, \sigma \in \Gamma(E),$$

$$D_{fV}\sigma = fD_V\sigma \quad \text{for } f \in C^\infty(M, \mathbb{R}), V \in \Gamma(TM).$$

(ii) $D$ is $\mathbb{R}$-linear in $\sigma$ :

$$D_V(\sigma + \tau) = D_V\sigma + D_V\tau \quad \text{for } V \in T_x M, \sigma, \tau \in \Gamma(E)$$

and it satisfies the following product rule:

$$D_V(f\sigma) = V(f) \cdot \sigma + fD_V\sigma \quad \text{for } f \in C^\infty(M, \mathbb{R}).$$

Of course, all these properties are satisfied for the differentiation of a vector field in $\mathbb{R}^d$ as described; in that case, we have $D_VX = dX(V)$.

Let $x_0 \in M$, and let $U$ be an open neighborhood of $x_0$ such that a chart for $M$ and a bundle chart for $E$ are defined on $U$. We thus obtain coordinate vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}$, and through the identification

$$E|_U \cong U \times \mathbb{R}^n \quad (n = \text{fiber dimension of } E),$$

a basis of $\mathbb{R}^n$ yields a basis $\mu_1, \ldots, \mu_n$ of sections of $E|_U$. For a connection $D$, we define the so-called Christoffel symbols $\Gamma^k_{ij}$ ($j, k = 1, \ldots, n, i = 1, \ldots, d$) by

$$D_{\frac{\partial}{\partial x^i}} \mu_j =: \Gamma^k_{ij} \mu_k.$$ 

We shall see below that the Christoffel symbols as defined here are a generalization of those introduced in §1.4.